MU120 Unit 7

The Open University

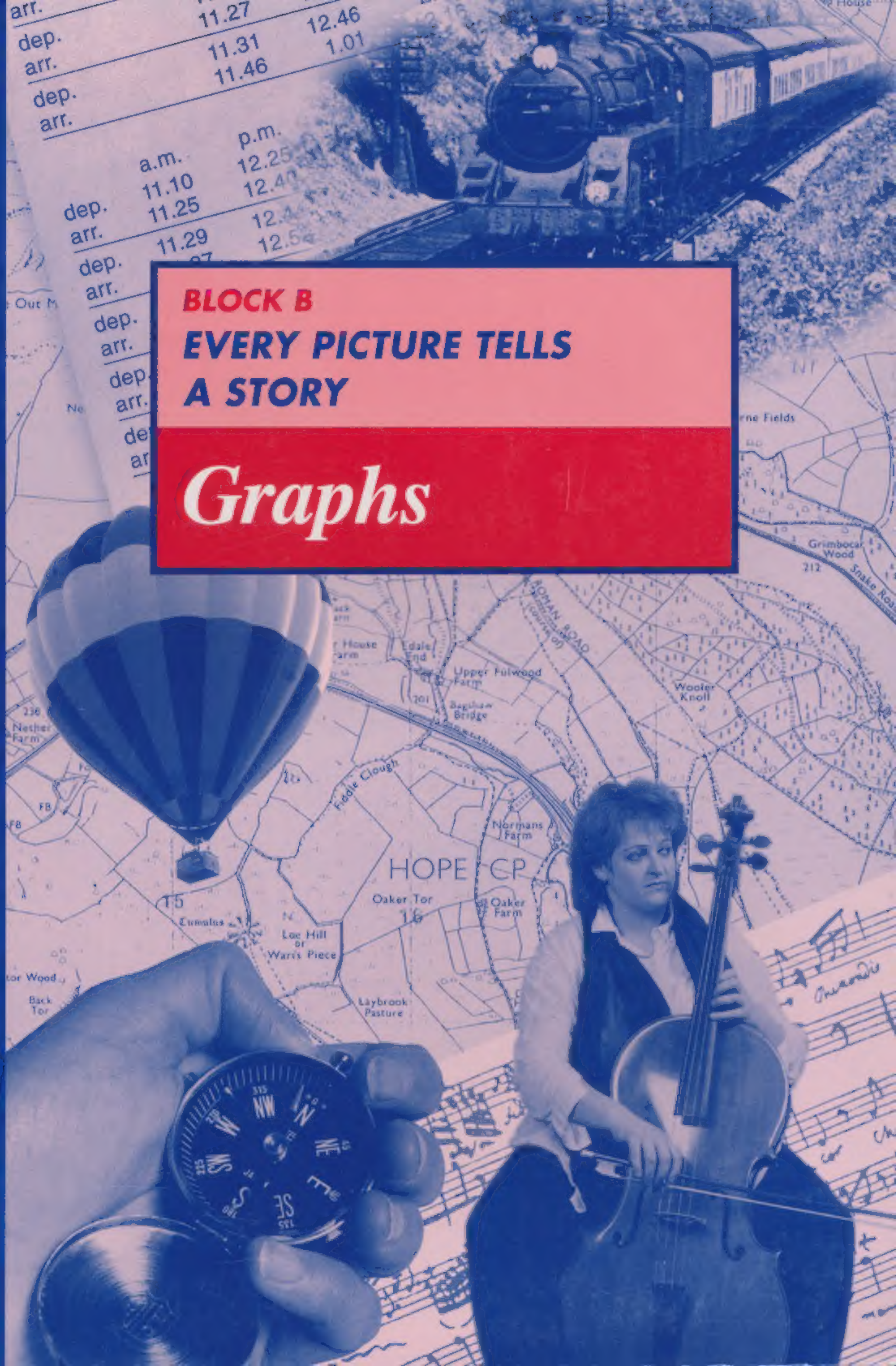
Mathematics and Computing
A first level multidisciplinary course

Open Mathematics

**BLOCK B**

**EVERY PICTURE TELLS A STORY**

# Graphs

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# *Graphs*

*Prepared by the course team*

MU120 course units were produced by the following team:

Gaynor Arrowsmith (Course Manager)
Mike Crampin (Author)
Margaret Crowe (Course Manager)
Fergus Daly (Academic Editor)
Judith Daniels (Author)
Chris Dillon (Author)
Judy Ekins (Chair and Author)
Barrie Galpin (Author and Academic Editor)
Alan Graham (Author and Academic Editor)
Linda Hodgkinson (Author)
Gillian Iossif (Author)
Joyce Johnson (Author)
Eric Love (Academic Editor)
David Pimm (Author and main Academic Editor)
Karen Rex (Author)

Other contributions to the text were made by a number of Open University staff and students and others acting as consultants, developmental testers, critical readers and writers of draft material. The course team are extremely grateful for their time and effort.

The course units were put into production by the following:

Course Materials Production Unit (Faculty of Mathematics and Computing)

Martin Brazier (Graphic Designer)
Hannah Brunt (Graphic Designer)
Alison Cadle (TEXOpS Manager)
Jenny Chalmers (Publishing Editor)
Sue Dobson (Graphic Artist)
Roger Lowry (Publishing Editor)
Diane Mole (Graphic Designer)
Kate Richenburg (Publishing Editor)
John A.Taylor (Graphic Artist)
Howie Twiner (Graphic Artist)
Nazlin Vohra (Graphic Designer)

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# Contents

# Study guide

The unit has four sections. It focuses on drawing and interpreting graphs. Sections 1 and 2 are about using graphical representations for different purposes. You will need to draw some graphs, so make sure you have a flat surface to work on, and you will need to have centimetre graph paper and your calculator to hand as you study the material. A short reader article is associated with Section 1.

Section 3 is based on a video band, 'Single-track minders'. The video lasts about twenty-five minutes, and at intervals you will be asked to stop the tape and tackle an activity. You should expect to spend up to about thirty minutes on some of the activities. You will need to set aside about three hours altogether for this study session. You may want to watch the video all the way through first, and then go back and watch the appropriate section again before trying the associated activity. These activities require you to draw graphs and make some calculations, so make sure you have some graph paper and your calculator with you at this point.

Section 4 is the calculator book section. At the end of this section, you will be asked to complete a Learning File and Handbook activity. You will find it useful to have completed your response to earlier activities to tackle this activity successfully. The activity may also be useful for part of the assessment of this unit.

*Graph paper is required for Sections 1, 2 and 3.*


Summary of sections and other course components needed for *Unit 7*

# Introduction

All representations (including graphical ones) rely on shared understandings of symbols and styles to convey meaning. Like maps, graphical representations stress some features and ignore others. As you work through this unit, bear in mind that graphs are selective representations of information. When you come across different graphs ask yourself what is being stressed and what is being ignored.

In the newspapers, you are likely to find graphs used to present all sorts of information: how the number of people who are jobless or homeless is changing; how interest rates and share prices are varying; how support for different political parties has changed.

It is easy to be seduced by graphs. They have an air of authority which defies you to challenge them. After all, many graphs used by the media have been drawn from data collected by reputable survey organizations. The graphs apparently simply show trends, variations, peaks and troughs as they are. However, no graph is simply a neutral representation of facts. All graphs are drawn from *some* point of view, drawn this way rather than that for a particular purpose. That is to say, someone has made some choices about how the graph should look. When you are faced with just a single graph, however, the choices may not be evident. One reason for knowing about a variety of types of graph, therefore, is to be more aware of what *could* have been drawn, when faced with what *was* drawn.

So an important message is that no graph is a value-free representation. Even if the intention is to present information in as unbiased a way as possible, that is itself a point of view. But some graphs will have been drawn specifically with the intention of presenting information in a particularly favourable or unfavourable light, to convince you of an argument or to influence your decisions. So take care when you are reading graphs that you are not misled or, perhaps more to the point, so that you do not mislead yourself. And when you are drawing your own graphs, you need to be clear about how you present your information to others so that your intentions are not misinterpreted or misunderstood.

As the American writer James Thurber put it: 'Get it right or let it alone. The conclusion you jump to may be your own.' (James Thurber (1956) *Further Fables for Our Time*, New York)

## *Activity 1* Organizing your study

Before you start on Section 1, take a few minutes to plan and organize your study of this unit. If it is helpful, use the printed planning sheet.

Remember to consider the assessment and your overall study plans for Block B. What aspects of your study technique do you want to concentrate on as a result of your own review, and the feedback from your tutor?

# 1 Every picture tells a story

*Aims* The main aim of this section is to give you practice in reading, interpreting and drawing a variety of graphs created for many different purposes. ◇

You will need graph paper for this section.

Graphs occur in all sorts of different contexts and applications. In *Unit 6*, you saw how graphical representations were used to show profiles of height plotted against distance for sections of the Peak District walk. This section looks at some other sorts of graphs: time-series graphs, conversion graphs and mathematical graphs.

A time-series graph shows how a measured quantity changes with time. This is one of the most common graphical forms. Time-series graphs can be used to look for trends in the way things change over time in order to predict what might happen in the future. Or they may indicate changes in trends which invite investigation and explanation.

Conversion graphs, as the name suggests, are drawn to give an easy way of converting between a quantity measured in one system of units, and the same quantity measured in another. So you can use conversion graphs to change unfamiliar units into ones you know, such as converting perhaps from degrees Celsius to degrees Fahrenheit, changing millilitres into fluid ounces, or converting miles into kilometres.

One of the contexts in which you will meet graphs is within mathematics itself. Graphs are used not only to show how physical quantities change with respect to each other—how the height of a hillside changes with distance, for example, or how a person's temperature changes with time—but are also used to represent mathematical relationships. And as you get used to the idea, you can use graphs to explore the mathematical relationships themselves.

You will find, however, that the language and conventions of graphs remain pretty much the same whatever the graph is used to represent.

The final subsection looks more critically at graphs and encourages you to begin to move from looking at a particular graph to looking *through* the graph: looking beyond the actual lines to raise questions about the choices that have been made in drawing the graph, to ask why a particular graph looks the way it does.

## Activity 2 Learning about graphs

You have just read a brief description of what this unit is about. Imagine you have been asked to explain to a small group of people involved in market research work how to draw a graph, and how to extract information from a graph. They need to have some knowledge of graphs

and graph drawing for their work, but some of them feel unsure about anything to do with graphs and numbers.

You have a wide variety of methods available to help you in your task. For example, you could use written instructions, activities, demonstrations, audio work and calculator exercises.

Plan which methods you might use to help this group and say why you would use those particular techniques.

Although you should make a start, you do not need to complete the activity at this point. You can come back to it again later. Towards the end of this unit you will be asked to review your choices.

## 1.1 Time-series graphs

> The time-series plot is the most frequently used form of graphic design. With one dimension marching along to the regular rhythm of seconds, minute, hours, days, weeks, months, years, centuries, or millennia, the natural ordering of the time scale gives this design a strength and efficiency of interpretation found in no other graphic arrangement.
>
> (Tufte, E. (1983) *The Visual Display of Quantitative Information*, Graphics Press, Connecticut, p. 28)

Graphs which show how a measured quantity varies as time changes are called time-series graphs. In a time-series graph, the horizontal axis represents the 'regular rhythm' of time in appropriate units and the vertical axis represents the quantity that varies over time.

Time-series graphs are favourites of the media, particularly newspapers. They are useful tools of comparative analysis, and are frequently used in support of particular arguments. Figure 1 comes from an article in a national newspaper relating changing levels of crime and drug abuse; Figure 2 appeared in an article critical of the government and shows the changing support (in 1995) for the three largest political parties in the UK; Figure 3 was found on the financial pages of a newspaper—it shows how exchange rates and stock market prices varied over three months.

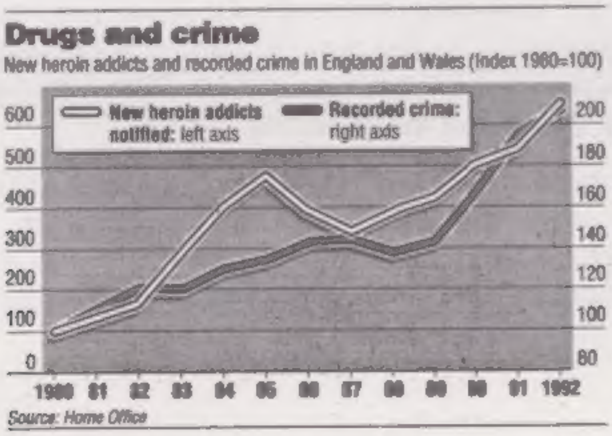



*Figure 1* Crime and drug abuse

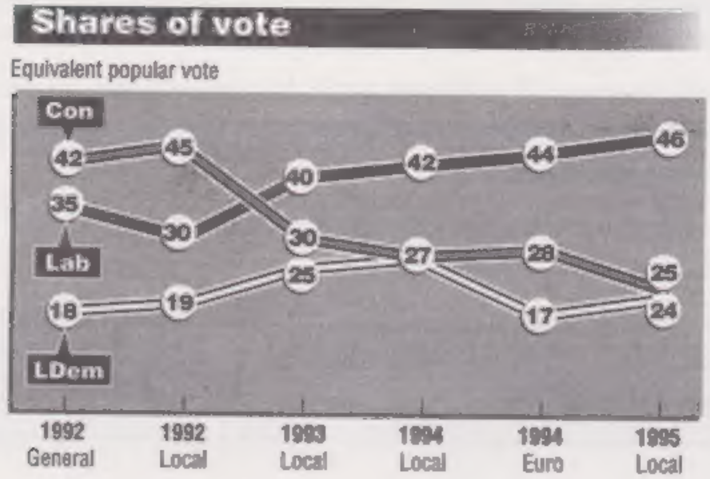



*Figure 2* Support for political parties

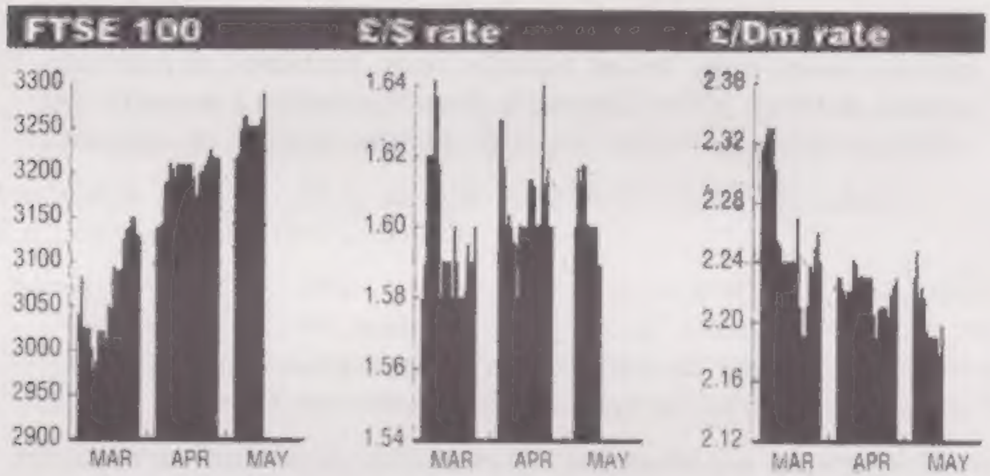
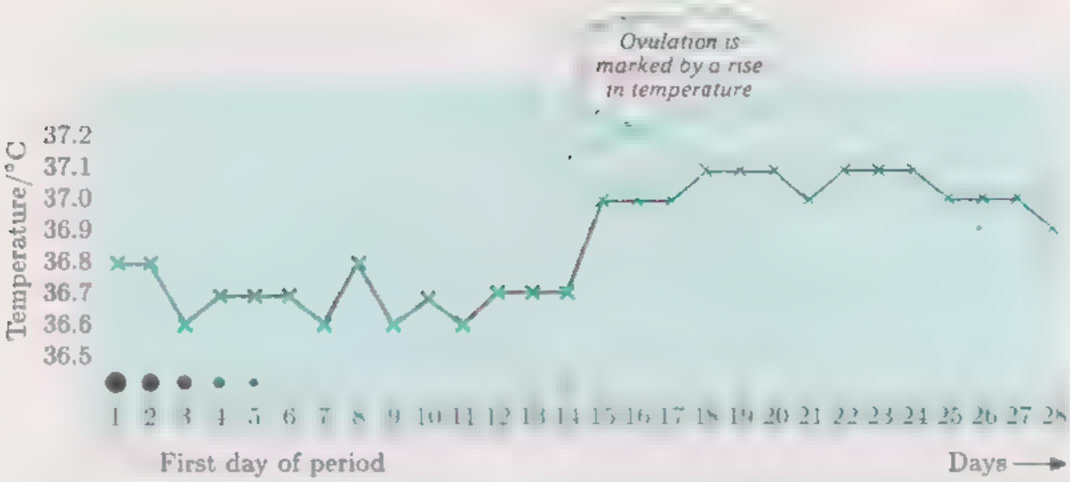



*Figure 3* Financial indicators

Time-series graphs should not be looked at in isolation from the context in which they appear. If a graph appears in a newspaper article, for example, it usually has a very specific role to play in supporting the argument being put forward by the writer. Remember that someone has chosen to include that particular graph; someone has chosen how the graph should look; and someone has chosen what information should be included on the graph and what should be ignored. Like the text in which it is embedded, the time-series graph does not appear by accident. It is there to reinforce visually the story the author wants to tell.

One use of time-series graphs is to keep track of how a quantity is changing so that an assessment or prediction can be made. In health care, for example, graphs of how a person's temperature varies with time can give indications both of the general health of the individual and act as a guide to specific physiological events.




This graph shows a typical variation; individuals will vary.

*Figure 4* Temperature over the monthly menstrual cycle

Figure 4 shows a time-series graph of a woman's temperature over her menstrual cycle, published in a pregnancy guide. Each point on the graph represents the temperature taken first thing in the morning. The points are joined by straight lines to give an overall visual indication of the temperature variation over the month. These lines contain no extra *information* about temperature, however. Since temperature is measured only once each day, the lines do not represent the woman's actual temperature during the intervening twenty-four hours, but instead reflect an assumption that the temperature does not vary wildly between measurements.

The important function of this graph is to show the way a woman's temperature *changes* over her monthly cycle. You can see that the temperature does not stay at the 'normal' body temperature of 37 °C, but varies from a minimum of about 36.6 °C at the beginning of the cycle to a peak of about 37.1 °C near the end, a range of about 0.5 °C. Since the variation is small compared with the average temperature, the vertical axis of the graph does not start from zero but covers only the temperature range that normally occurs.

Conversion between degrees Fahrenheit (°F) and degrees Celsius (°C) is discussed in Subsection 1.2.

Plotting the graph at this scale shows up variations as small as 0.1 °C, which would be lost if the scale covered a wider range. At this level of detail, the time-series graph shows quite clearly the rapid rise (relatively) in temperature that signals that ovulation has taken place. Since a causal link between ovulation and temperature rise has been established by medical research, a woman plotting her daily temperature as a time-series graph can use it to determine when ovulation has occurred. Thus, she can *read out* the temperature rise from the graph, and bring other knowledge to bear to *read in* the interpretation that an egg has been released.

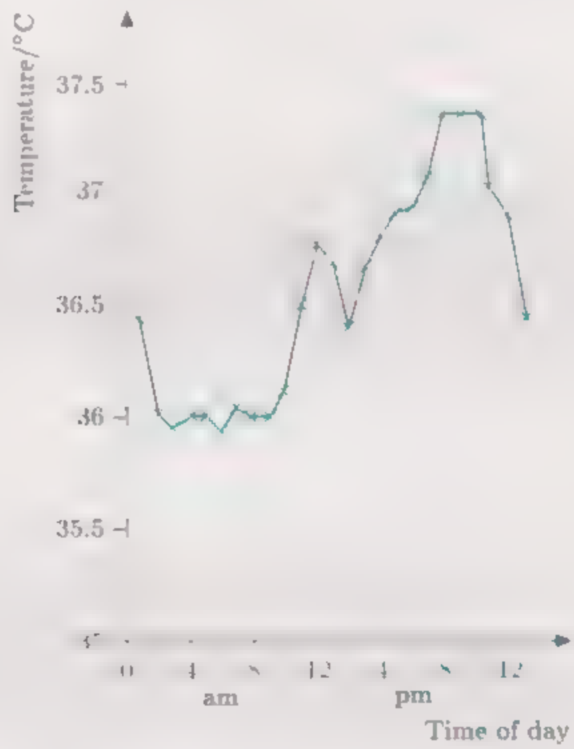
Recall the discussion about 'reading out' and 'reading in' from *Unit 6*. Here the graph is treated as a complex symbol which can be interpreted to convey meaning.

This interpretation of the graph also relies on the accuracy with which the temperature readings have been taken and plotted. In this case, the thermometer must be a special 'fertility' thermometer which will give readings to an accuracy and resolution sufficient to respond to the relatively small changes taking place. If the measurements were accurate

only to within say, 0.5 degrees, or if the graph scale itself did not allow you to plot points any more accurately than to within 0.5 degrees, then the fine detail of the temperature variations would be lost and a relatively reliable indication of ovulation would be less likely.

Time-series graphs give information only at the points that have been explicitly plotted. Even if a line has been drawn joining the plotted points, it can, at best, only represent an informed guess of what is going on between the points. A lot of mathematical effort has gone into developing techniques both for *interpolation* — estimating the values that lie between known points — and *extrapolation* — estimating values that lie beyond the range of plotted data. But whatever assumptions are made about long- and short-term trends, the measured data are all there is to go on. This point is brought out by Figure 5.

*Trends* are expressions of confidence that a set of data conforms to some recognizable pattern, rather than being just a set of random and unrelated numbers.




Again this is a typical variation.

*Figure 5* Temperature variation over a day

The daily, or *diurnal*, temperature variations are similar in men and women.

But notice that the notion of 'rapid' and the visual perception of steepness is dramatically altered by the scale chosen for the vertical axis.

Temperature data can be plotted as a time series, but this time the graph characterizes the daily rather than the monthly rhythm of a woman's temperature variation. After waking, a woman's temperature rises rapidly, followed by a dip in the afternoon, a gradual climb to a peak in the evening and then a sharp drop during sleep. The variation over a day is about 1.5 °C, significantly greater than the monthly variation. In between the adjacent points on the monthly graph, therefore, there is a lot of hidden activity.

So there is a trade-off. The time-series graph based on a single daily measurement gives no indication of a daily cycle, particularly as it was deliberately taken at the same time each day. On the other hand, readings taken throughout the day will give a more detailed record, but the extent

of the daily fluctuations may obscure the single small temperature rise that signals ovulation. Because of the variation of temperature over the course of a day, temperature measurements must be taken at the same time each day for the monthly series to be meaningful. Taking a morning reading one day and an afternoon reading the next could indicate a misleading temperature rise. Similarly, temperatures quoted on their own can be a misleading guide to a woman's general state of health; a temperature higher than 37 °C may indicate a slight fever if it is recorded in the morning, but not if it is recorded in the evening or after ovulation. This is, of course, one of the difficulties—body temperature rises for a variety of reasons.

So time-series graphs must be read with care. Adopt a questioning attitude when you are faced with a graph. Look carefully at the vertical axis to see just what the range of variation is, and at the horizontal axis to see what time intervals have been chosen. Ask yourself about the significance of this choice—what might be going on between each plotted point?

You might question whether the plotted variation is significant or whether it is the result of expected fluctuations. What about the accuracy of the figures, and the accuracy with which they have been plotted? Look at the line of the graph itself, if there is one. Points will not always be joined by straight lines—ask yourself what assumptions have been made about the progression from one time point to the next.

## Activity 3 Study patterns

In 1916, a study conducted on a class of psychology students at the University of California revealed a variation of study efficiency over the day. A group of 165 students were asked to give their preferred hours of study, and then undertook five repetitive memory tasks at one-hour intervals over three consecutive days. The tasks tested their ability to remember short sequences of numbers, to substitute numbers for symbols, to recognize geometrical figures, and to remember simple ideas.

Mathematical tasks are common choices for psychological experiments.

A composite measure of efficiency was calculated for the group each hour. The results are given in Table 1. Column 1 gives the time of day, column 2 gives the corresponding relative efficiency measure, and column 3 lists the numbers of students preferring to study at that time of day.

Using your calculator, display the study efficiency data as a time-series [illegible]

[illegible]

your calculator.

Look at Chapter 6 of the *Calculator Book* if you need a reminder about how to display graphs from tables of data.

What conclusions can you draw from these graphs about the study patterns of the students? Would your own study pattern fit into this picture? Note down your response.

*Table 1* Preferred hours and study performance for the student group

| Time of day | Relative study efficiency | Study time preference |
| --- | --- | --- |
| 08.00 | 100 | 110 |
| 09.00 | 104.2 | 150 |
| 10.00 | 106.7 | 90 |
| 11.00 | 105.5 | 40 |
| 12.00 | no data | no data |
| 13.00 | 98.5 | 5 |
| 14.00 | 100.8 | 4 |
| 15.00 | 105 | 4 |
| 16.00 | 104 | 15 |
| 17.00 | 100.1 | 15 |

(Based on Palmer, J. D. (1976) *An Introduction to Biological Rhythms*, Academic Press, London, p. 142)

Here you have been reading about graphs, you were asked to consider ways of making interpretations from them and have used your calculator to display a graph. Look back to Activity 2 and note briefly which methods you feel are most useful in learning about graphs.

## 1.2 Graphical conversions

The term 'conversion graph' describes a graph used to convert a quantity measured in one system of units to the same quantity measured in another. For example, you can draw up a conversion graph to convert temperatures expressed in degrees Celsius to temperatures expressed in degrees Fahrenheit; to convert liquid volumes expressed in pints to the same volumes expressed in litres; to convert a sum of money expressed in one currency to the same amount expressed in a different currency.

In *Unit 6*, you met the idea of using mathematics to convert one measurement into another. You used the map scale to convert between distances and areas on the map and on the ground. The relationships you used have the general form

$$\text{measurement on the ground} = \text{some number} \times \text{measurement on the map.}$$

For your 1 : 25 000 map, the relationships between distances and areas are

$$\text{distance on the ground} = 25\,000 \times \text{distance on the map}$$

and

$$\text{area on the ground} = (25\,000)^2 \times \text{area on the map}$$

where, in each case, the map and ground measurements used the same units.

Now these relationships are expressed as formulas, but you can represent the same information graphically. Representing things in a different way can offer a new perspective and a new way of thinking about a relationship. Often it is helpful to move between different viewpoints and

different representations when you are trying to understand a problem. Figure 6 shows the relationship between distances in the form of a graph.

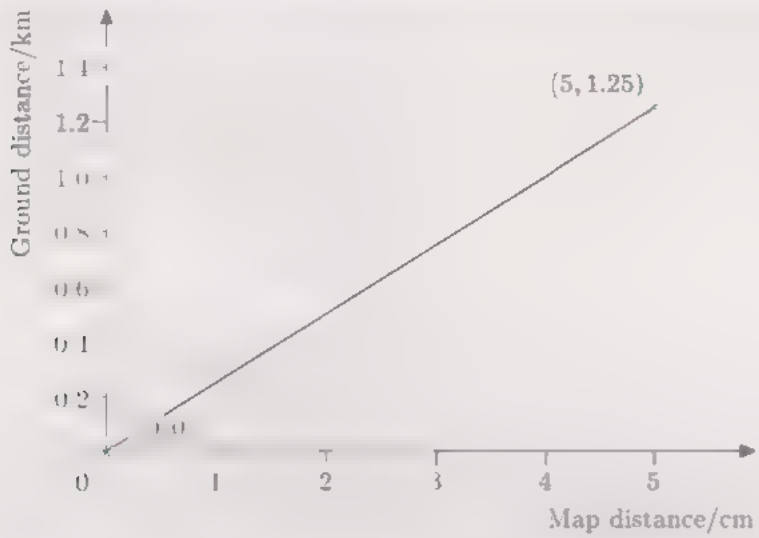



*Figure 6* Graph of ground distance against map distance

Look at the way Figure 6 has been constructed. By convention, the map distance—which is what you know—is plotted along the horizontal axis, and the ground distance—which is what you want to find out—is plotted along the vertical axis. Notice the scales and the labels on the axes. Map distances are conveniently measured in centimetres, but ground distances are more conveniently quoted in kilometres. So in this case the scale on the horizontal axis relates to a measurement in centimetres and the scale on the vertical axis relates to a measurement in kilometres.

The axis of a graph is rather like a map. The axis represents a quantity (such as temperature, distance, time). The scale of the axis, like the scale of a map, relates the distance along the axis from the origin to the amount of the quantity.

The labels on each axis tell you about the thing that is measured and the units it is measured in. So the horizontal axis is labelled 'Map distance/cm' and the vertical axis is labelled 'Ground distance/km'. Notice that there is an oblique line between the quantity and its units: ground distance divided by kilometres, and map distance divided by centimetres. This means that the numbers along the horizontal and vertical axis are ratios—pure numbers—which can be added, subtracted, multiplied or divided without worrying about their units.

Remember that quantities with units are the result of measurements taken in the real world. When you draw a graph you are not dealing with actual kilometres or centimetres but with their representations as lengths along the axes of the graph. You can perform calculations with the numbers to make predictions about the material world. But when you go back to that world, remember to restore the units, so that the numbers relate once again to actual measurable distances.

Returning to the formula for the conversion graph, recall that the starting point is a scale relationship which links map distance to ground distance, when both are measured in the same units. If the units are centimetres, you have

$$\text{ground distance in centimetres} = 25\,000 \times \text{map distance in centimetres}$$

But how to convert from a map distance measured in centimetres to a ground distance measured in kilometres? This changes the formula slightly.

There are 100 000 centimetres in a kilometre, so to convert from centimetres to kilometres divide by 100 000. The formula becomes:

$$\begin{aligned}\text{ground distance in kilometres} &= \frac{\text{ground distance in centimetres}}{100\,000}\\ &= \frac{25\,000 \times \text{map distance in centimetres}}{100\,000}\\ &= 0.25 \times \text{map distance in centimetres}\end{aligned}$$

Whatever the map distance is in centimetres, the ground distance in kilometres can be found by multiplying by 0.25. Mathematically, the ground distance is said to be *directly proportional* to the map distance. This means that ground distance is found simply by multiplying the corresponding map distance by some fixed number, known as the *constant of proportionality*. Another way of expressing this is the fact that the ratio of ground distance to map distance is constant (and equal to this constant of proportionality).

ground distance (in km)/map distance (in cm) = 0.25

This means, for instance, that if you double one value, the effect is to double the other, and if you third one value, the upshot is that the other is divided by three as well. And the fact that the relationship is directly proportional has an important consequence for the graph—it will necessarily be a straight line.

This is the same for any distance or length conversion (and others such as area, or weight, or volume).

Only two points are needed to draw a straight-line graph. Choosing one of the points is straightforward, it is the origin of the graph. Why? Because zero distance on the map corresponds to zero horizontal distance on the ground. So one point of the graph here must have the coordinates $(0, 0)$. But this will not always be where the two axes meet. Remember from Figure 1, that not all graphs are drawn with the vertical axis scale starting at zero.

The other point can be chosen so that it fits conveniently into the range you want the graph to cover. Figure 6 shows map distances up to 5 cm, corresponding to ground distances up to $0.25 \times 5 = 1.25$ km. So the second point can be placed at the top end of the scale at $(5, 1.25)$. In fact, the further the second point can be placed from the origin the better, because inaccuracies in drawing the graph are reduced.

Look at Figure 7(a). Here the second point is very close to the origin. Any inaccuracy in plotting this point or in drawing the line through the point will be magnified significantly at the end of the range. As a result, the graph will become less accurate the further you move from the origin. If the second point is put as far away as possible, as in Figure 7(b), any drawing inaccuracy will result in lower errors over the range of the graph.

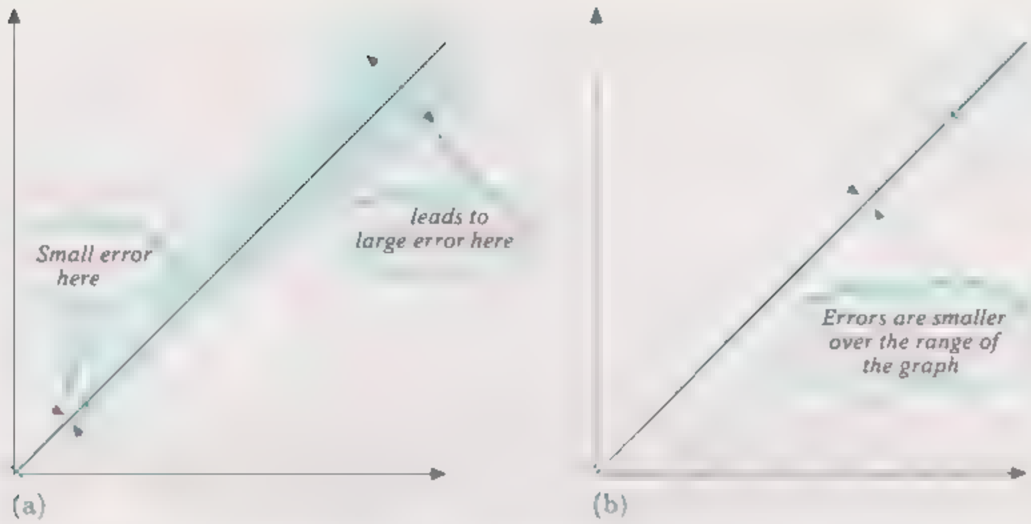



*Figure 7* Effect on accuracy of the position of the second point of a straight line graph

A straight line links the points (0, 0) and (5, 1.25) on Figure 6 and represents the proportional relationship between map and ground distances.

▶ How do you use the graph?

Look at Figure 8. Start with the map distance on the horizontal scale, move vertically up until you reach the line, then move horizontally until you reach the vertical axis. The number at that point will give you the corresponding ground distance in kilometres.

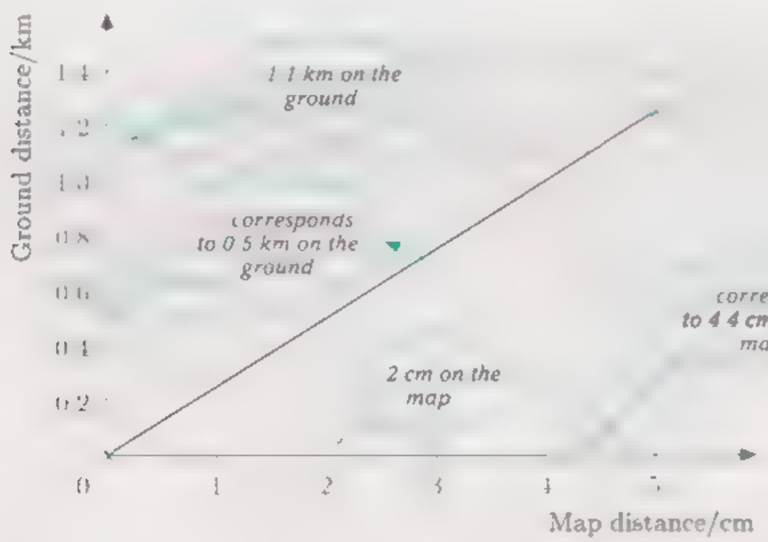



*Figure 8* Converting between map distances and ground distances

You can also use the graph to go from ground distance to map distance. Find the particular ground distance on the vertical scale, move horizontally across to the line and then move vertically down until you

reach the horizontal scale. You can then read off the corresponding map distance in centimetres. If you could be bothered to do it—and had unlimited time (like forever!)—you could have drawn the same graph by working out every possible ground distance for every possible map distance between 0 and 5 cm, and plotting each coordinate pair. But the results of all those countless, individual, specific calculations are automatically included in just one straight line drawn using the knowledge that the relationship is a directly proportional one. A mathematical formula generalizes a relationship by containing markers—words or symbols—which you replace by numbers for specific calculations. Likewise, a graph generalizes the results of individual calculations, indicating by means of its shape the corresponding values in a relationship. A graph, therefore, represents the general form of a directly proportional relationship as well as allowing you to handle specific examples.

There is much more on proportional relationships in *Unit 13*.

## Activity 4 *A litre of water . . .*

When the metric system of weights and measures was introduced in the UK, the government information campaign included the rhyme: 'A litre of water's a pint and three-quarters', to help people remember the conversion factor. The conversion constant is not specific to water (it merely helped the rhyme) and can be used for any fluid.

(a) With the campaign slogan in mind, draw a graph on graph paper to convert between pints and litres. Use your graph to find (a) how many litres correspond to 3 pints, and (b) how many pints correspond to 4 litres.

(b) Using the fact that there are 8 pints to 1 gallon, find the conversion factor between gallons and litres, and draw up a table of gallon/litre equivalents. Do you think it would be more helpful if you used a conversion graph rather than a table? Write down any advantages or disadvantages you can think of.

Recall that two quantities being directly proportional to each other—such as map and ground distances, or pints and litres—means the amount of one is found simply by multiplying the other by a fixed conversion number. In Activity 4, for example, the formula describing the relationship (according to the rhyme) is:

$$\text{volume in pints} = 1.75 \times \text{volume in litres}.$$

The number 1.75 is the constant of proportionality. It relates a volume measured in pints to the same volume measured in litres, and strictly it has units itself. In this case, the units are 'pints per litre'.

▶ How is the constant of proportionality represented on a graph?

One of the main features of a straight-line graph is that the line has a constant slope. The gradient of the slope is numerically equal to the constant of proportionality. Look at Figure 6 to see what this means for the map example. For a 1 : 25 000 map, the constant of proportionality between ground distances in kilometres and map distances in centimetres is 0.25 km per cm. So the gradient of the corresponding graph is 0.25.

A similar relationship holds for a 1 : 50 000 map. In this case, 1 cm on the map corresponds to 0.5 km on the ground, so the constant of proportionality is 0.5 km per cm and the gradient of the corresponding graph is 0.5. The steeper gradient says, in effect, that you get more kilometres for your centimetre on this map.

In general, therefore, if the line passes through the origin of a straight-line graph, then the gradient of the graph links the values on the horizontal and vertical axes. The relationship is:

value on vertical axis = gradient × value on horizontal axis.

Different constants of proportionality give straight-line graphs with different gradients. The steeper the gradient, the greater the value on the vertical axis for a given value on the horizontal axis. Recall the earlier comment in *Unit 6* about the effect of changing the scale on the vertical axis on the visual perception of steepness. Choice of scale can have a profound effect on the visual impact of a graph. But *numerically*, the gradient of the graph is unchanged by a simple change of axis scale.

## Activity 5 Converting to metric

If you were to draw graphs to convert from (a) pounds to kilograms (1 pound = 454 grams), or (b) miles to kilometres (1 km = 0.621 miles), what would be the gradient in each case?

Make a quick sketch of the conversion graph in each case.

Did you find making the sketches helped you to answer the question? Or did you find they made the task more difficult? Make a few brief notes to record your response.

For many measures like those in Activity 5, a conversion graph will be a straight line starting at the point (0, 0), because zero on one scale of measurement will also be zero on another. But this is not true for all measurement scales. Temperature, for example, can be measured using different scales which do not share the same zero point, because a temperature of zero degrees can be defined in different ways. Zero degrees Celsius, for instance, does not mean 'no heat'.

For everyday use, most people tend to think in terms of either the Fahrenheit or the Celsius (or centigrade) scales. Which do you use? When you hear temperatures given in a weather forecast in degrees Celsius

(written as °C), do these mean much to you, or do you try to get a feel for how hot or cold it is going to be by converting to degrees Fahrenheit (written as °F)? Older cookery books and ovens often quote temperatures in degrees Fahrenheit, whereas modern ones use degrees Celsius. How would you convert between the two temperature scales? What mental picture of the two scales would you use?

### *Celsius and Fahrenheit*

The Celsius, or centigrade, temperature scale is named after Anders Celsius, a Swedish scientist who first devised a form of this scale in 1742. The word 'centigrade' means 'one hundred steps', and refers to the fact of expressly choosing 100 equal divisions between the boiling and freezing points of water.

Daniel Fahrenheit (1686–1736) was a German scientist who developed the first thermometers using the expansion of alcohol and mercury to indicate temperature.

The Celsius scale is defined in terms of the freezing and boiling points of water (at a particular standard air pressure). When water starts to freeze and form ice, its temperature is *defined* to be 0 °C; when it boils and forms steam, its temperature is *defined* to be 100 °C. Thermometers are calibrated using these fixed points: they are made so that they read 0 °C when they are immersed in freezing water and 100 °C when they are immersed in boiling water.

The Fahrenheit scale was originally set up by taking 0 °F as the freezing temperature of a mixture of ice, water and salt, and 96 °F (later adjusted to 98.6 °F) as the 'normal' temperature of the human body. On this scale, pure water freezes at 32 °F and boils at 212 °F.

### *Kelvin*

In science, temperatures are often quoted using the absolute or Kelvin scale on which zero (the so-called *absolute* zero, the lowest temperature theoretically possible) corresponds to about −273 °C, or about −460 °F. Here, zero on the Kelvin scale does mean 'no heat'.

▶ How would you go about drawing a graph to convert from one scale to the other?

First you need some data about corresponding temperatures on each scale. In the case of Celsius and Fahrenheit, there are two fixed points of reference: the freezing and boiling points of water. On the Celsius scale, the freezing point is defined to be 0 °C; on the Fahrenheit scale, the freezing point is 32 °F. So if you plot degrees Celsius on the horizontal axis and degrees Fahrenheit on the vertical axis of a graph, the freezing point of water is represented by a point with the coordinates (0, 32).

You can also relate the corresponding values at which water boils. On the Celsius scale, 100 °C is defined to be the temperature at which boiling

occurs; on the Fahrenheit scale, boiling occurs at 212 °F. So on the graph the boiling point of water is represented by the point (100, 212).

The two fixed reference points are plotted in Figure 9 and joined by a straight line. This graph enables you to convert any temperature value between 0 and 100 °C on the Celsius scale to the corresponding temperature value between 32 and 212 °F on the Fahrenheit scale, and vice versa.

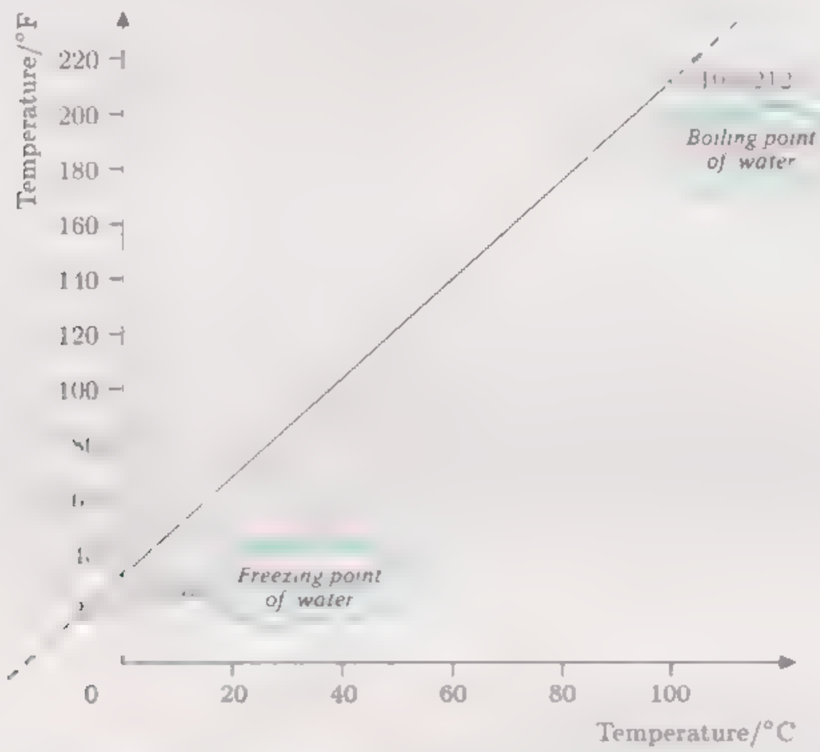



*Figure 9* Temperature conversion

## *Activity 6* Temperature conversion

Use the graph in Figure 9 to find the values corresponding to 15 °C and 200 °F.

If you extend the graph above and below the reference points you can convert from any temperature value on one scale to the corresponding value on the other. This is useful if you want to extend the range of the graph, so that you can use it for temperatures that are below the freezing point, or above the boiling point, of water.

Recall, extending a graph beyond the known values is called 'extrapolation'. How do you know that the extrapolated graph will continue to be a straight line for all temperatures on the two scales? It will be so, because the relationship between the Celsius and Fahrenheit scales is not a matter for experiment, it is *defined* to be a straight line for all temperatures.

Figure 9 shows the range of temperatures appropriate for the liquid state of water. However, would this be a good range for the graph if you were drawing a conversion graph for normal air temperatures in Europe?

Air temperatures in Europe rarely go above 50 °C, so there is no point in going as high as the boiling point of water. However, they do go below freezing point. So a range of −50 °C to 50 °C might be a good starting point.

## *Activity 7* *Changing the range*

By extending the straight line on Figure 9 below 0 °C, draw a graph that you can use to convert from Celsius to Fahrenheit over the range −50 °C to 50 °C. What is the corresponding range of the Fahrenheit scale?

Is there any point for which the numerical values on each scale are the same? If you extended the line indefinitely in both directions, how many other such points do you think you would find?

---

A temperature conversion graph is different from the earlier conversion graphs. The line does not start at the point (0, 0) and the relationship between degrees Celsius and degrees Fahrenheit is not a direct proportionality. For example, a temperature of 15 °C corresponds to 59 °F, but a temperature 30 °C does not correspond to 2 × 59 °F = 118 °F. In fact, it corresponds to 86 °F. Doubling the temperature on the Celsius scale, therefore, is not equivalent to doubling the temperature on the Fahrenheit scale, and vice versa. In other words, you *cannot* express the relationship between the scales in the form

degrees Fahrenheit = some number × degrees Centigrade

▶ So what is the relationship between the two scales?

You know that if the line passes through the origin of a straight-line graph, then the gradient of the graph links the values on the horizontal and vertical axes. The general relationship is:

value on vertical axis = gradient × value on horizontal axis

Now suppose the scale on the vertical axis of Figure 9 is changed by subtracting 32 from each number to produce a new scale. This is illustrated in Figure 10. The Fahrenheit scale and the new scale are very simply related by the formula:

new scale = Fahrenheit scale − 32

This change of scale on the vertical axis does not affect the gradient.

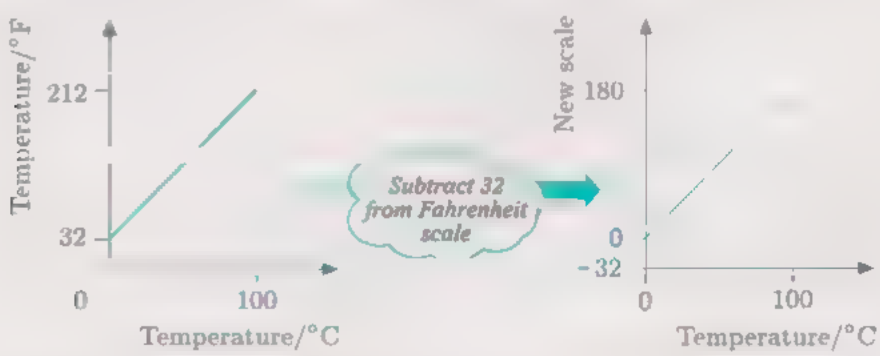



*Figure 10* Subtracting 32 from the vertical scale

You can see that, whereas the Fahrenheit scale goes from 32 to 212, the new scale goes from 0 to 180. Now a new straight line graph can be drawn, as in Figure 11, which goes through the point (0, 0) at one end and the point (100, 180) at the other. You can work out the directly proportional relationship between the Celsius scale and the new scale by finding the gradient. The relationship is:

value on new scale = gradient × value on Celsius scale.

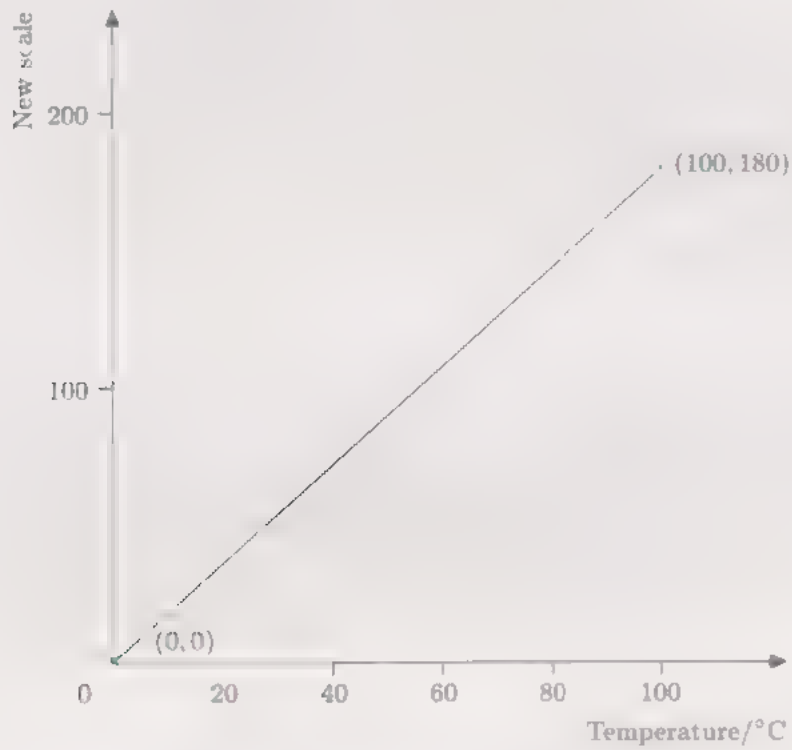



*Figure 11* A directly proportional relationship

▶ What is the gradient of the straight line in Figure 11?

The new vertical scale goes from 0 to 180 as the Celsius scale on the horizontal axis goes from 0 to 100. So the gradient is

$$\frac{\text{vertical change}}{\text{horizontal change}} = \frac{180}{100} = 1.8$$

and the relationship between the scales on the axes is:

value on new scale = 1.8 × value on Celsius scale. (1)

The formula tells you that there is a directly proportional relationship between the value on the Celsius scale, drawn on the horizontal axis, and the value on the new scale, drawn on the vertical axis. But this is not quite what was wanted.

▶ What is the relationship between the Fahrenheit and the Celsius scales?

You can get the Fahrenheit scale back simply by adding 32 to the new scale, that is:

Fahrenheit scale = value on new scale + 32. (2)

Now you need something involving the Celsius scale on the right-hand side of formula (2). Notice that the words 'value on new scale' appear in both formula (2) and formula (1). Formula (1) deals with the Celsius scale and relates it to the new scale. The equals sign in formula (1) tells you that the words 'value on new scale' and '1.8 × value on Celsius scale' both refer to the same number.

Referencing formulas by numbers in brackets like (1) is part of the conventional style of the 'written language' of mathematics. It allows you to refer back to particular ones easily.

The words 'value on new scale' in formula (2) also refer to this number. Where you get different words or symbols referring to the same thing, you can replace one by another in a formula without changing the formula's numerical value. So you can replace the words 'value on new scale' in formula (2) by the expression '1.8 × value on Celsius scale' from formula (1) to get a new formula:

$$\text{Fahrenheit scale} = 1.8 \times \text{Celsius scale} + 32 \tag{3}$$

This is the relationship you are looking for between values on the Celsius and Fahrenheit scales: the formula that represents the straight line graph in Figure 12. You may have come across this relationship in the form of the rule 'to convert from Celsius to Fahrenheit, multiply by 9 over 5 and add 32'. 9/5 is, of course, equal to 1.8.

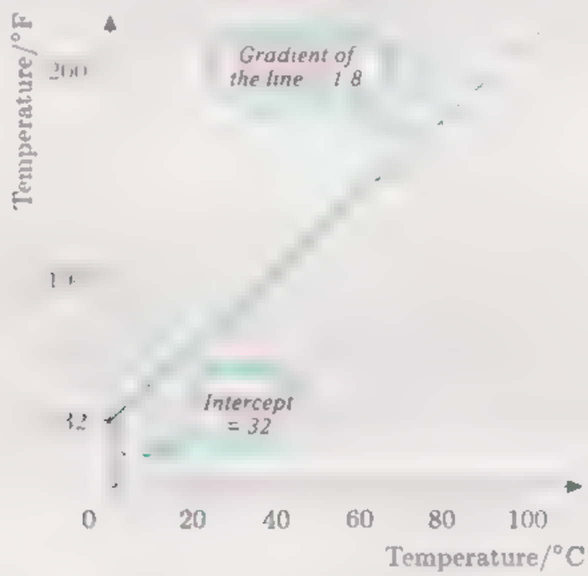



*Figure 12* Graph of 'Fahrenheit = 1.8 × Celsius + 32'

You can see from Figure 12 that the graph crosses the vertical axis at 32. This number also appears in formula (3). It is called the *intercept* of the graph (it is where the graph *intercepts* the vertical axis). The number 1.8, also appearing in formula (3), is the gradient of the graph.

Here it is the $y$-intercept.

## *Activity 8* Cooking times

A cookery book suggests that the cooking time for chicken in an oven preheated to 180 °C is calculated by allowing 20 minutes for each $\frac{1}{2}$ kg, plus a further 20 minutes.

(a) What is the formula relating the cooking time in minutes to the weight in kilograms?

(b) On squared paper, draw a graph to give cooking times in minutes for weights up to 3 kg. What is the intercept and the gradient of the graph? Does this graphical relationship make sense for *all* weights?

Section 1 of *Unit 1* discussed testing mathematical relationships by pushing them to extremes.

A book on microwave cooking suggests that the cooking time for chicken is simply 16 minutes for each kilogram. Draw this graph on the same graph paper and use your graphs to find the cooking times for a 1.6 kg chicken in a conventional and a microwave oven.

Do not forget to scale and label the axes, and give the graph a title.

---

This subsection started by looking at conversion graphs which were straight lines passing through the origin of the graph. The intercept in those cases was zero, and only one number—the gradient—was needed to describe the relationship between the quantities plotted on the horizontal and vertical axes. In the more general case, the graph is still a straight line with a constant gradient, but the line no longer goes through the origin. **An extra number—the intercept—is used to pin the graph down to a** particular location. You can think of a straight line graph with a fixed slope being able to move vertically up or down, as in Figure 13. You can see that moving the graph vertically upwards increases the intercept, while moving it in a downwards direction decreases the intercept. If the straight line crosses the vertical axis below zero, the intercept is a negative number.

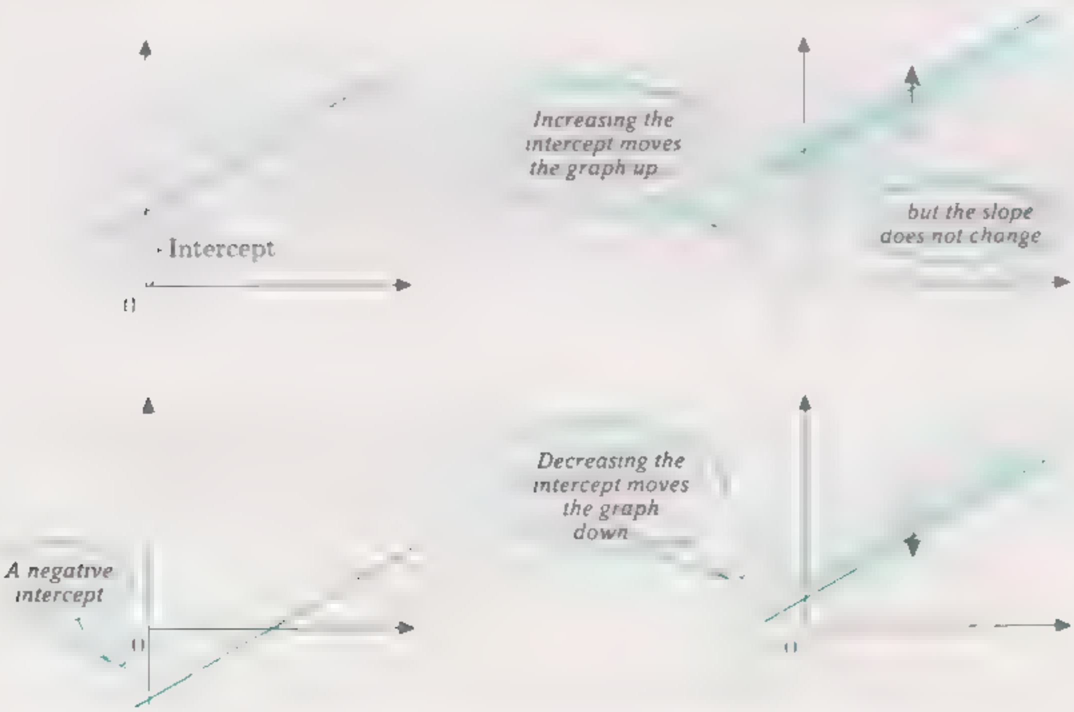



*Figure 13* Effect of changing the intercept on a straight-line graph

The general straight-line graph is described by the following formula:

$$\text{value on the vertical axis} = \text{gradient} \times \text{value on the horizontal axis} + \text{intercept}$$

In the examples of conversion between different units you have seen this relationship in two forms:

- where one quantity is directly proportional to another. The intercept is zero and the graph passes through the point $(0, 0)$;
- where the quantities are related by a straight line, but zero in one system of units does not coincide with zero in another. The graph does not pass through the point $(0, 0)$. The formula includes a second fixed number—the intercept—which is the value at which the straight line meets the vertical axis.

## *Activity 9* Straight-line models

Spend a few minutes to note down some examples of relationships you think can be represented by straight-line graphs. Think about conversions you might make between different sorts of measurements or quantities, think about how bills are worked out. Which are directly proportional relationships and which are not?

Recall the electricity bill example from Section 1 of *Unit 1*.

## 1.3 Mathematical graphs

Mathematicians use some special terms to talk about graphs. Understanding and feeling confident with this graphical language is as much a part of mathematics as doing calculations, or working with formulas. By convention the horizontal axis of a graph—the one running across the page from left to right—is often called the '$x$-axis', and the vertical axis—the one running up the page—is called the '$y$-axis', as in Figure 14. As a reminder, the $x$-axis and the $y$-axis are often simply labelled '$x$' and '$y$', respectively.

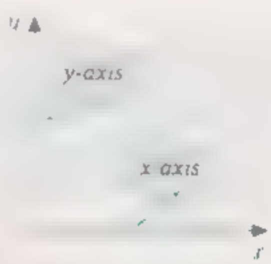



*Figure 14* The $x$- and $y$-axes of a graph

When only positive quantities are plotted, the two axes are conventionally drawn on the left-hand and bottom edge of the graph. However, you might want to plot negative values as well. As Figure 15 shows, both axes can be

extended in a negative as well as a positive direction. This follows a mathematical convention about representing numbers as points on a line: on the $x$-axis, positive numbers increase to the right and negative numbers increase to the left. Similarly, on the $y$-axis, positive numbers increase up the page, and negative numbers increase down the page.

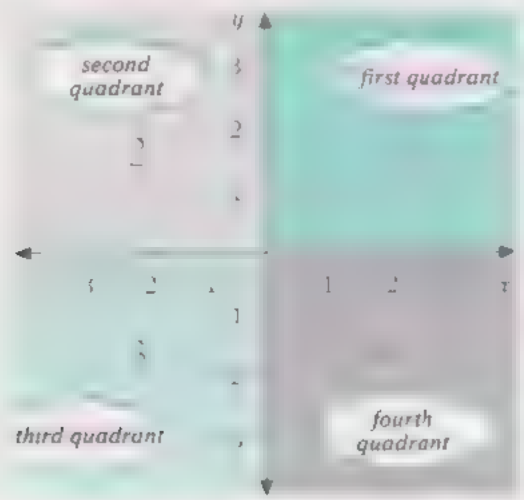



*Figure 15* Positive and negative axes

Using negative as well as positive axes divides a graph up into four regions, called 'quadrants'. In Figure 15 the quadrants are numbered from 1 to 4. The convention is that the numbering goes in an anticlockwise direction. In the first quadrant, the $x$-axis and $y$-axis both represent positive values. In the second quadrant, the $y$-axis values remain positive but values along the $x$-axis are negative. In the third quadrant, both the $x$- and $y$-axes mark negative values. In the fourth quadrant, values along the $x$-axis are positive while the $y$-axis values are negative.

The prefix 'quadr-' as in *quadrilateral*, *quadrangle* and *quadrant* indicates four-ness.

Recall that each point on a graph is represented by a pair of numbers called *coordinates*. Look at Figure 16. The position of the point measured along the $x$-axis is called, not surprisingly perhaps, the *$x$-coordinate*, and the position measured along the $y$-axis is called the *$y$-coordinate*.

In some mathematics books you may come across the terms 'abscissa' for the $x$-coordinate and 'ordinate' for the $y$-coordinate.

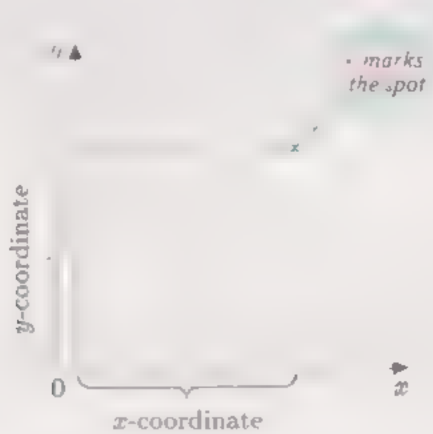



*Figure 16* The coordinates of a point

Coordinates which locate a point by referring to its position relative to two (or three) axes intersecting at right angles are called Cartesian coordinates.

The coordinates of a point are always given in the form

(value along the $x$-axis, value along the $y$-axis).

Unlike an Ordnance Survey grid reference which refers to a small square region on a map, graph coordinates refer only to single points.

Two values separated by a comma and enclosed in round brackets form a coordinate pair. Figure 17 shows how the values of the coordinates specify points in the different quadrants. $(3, 2)$ is a point in the first quadrant. Its position is specified by moving three units horizontally along the $x$-axis followed by two units vertically, parallel to the $y$-axis.

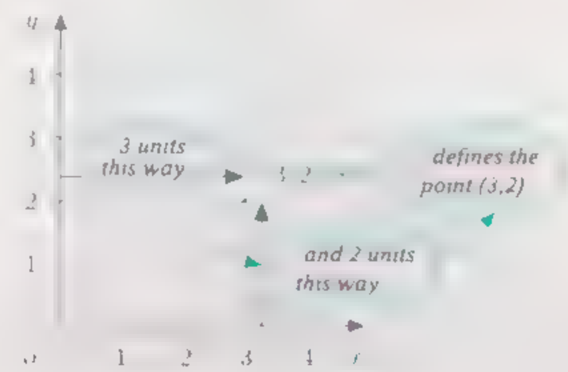



*Figure 17* Plotting coordinate pairs

Similarly, in Figure 18, $(-2, 1)$ specifies a point in the second quadrant, $(-3, -3)$ is a point in the third quadrant and $(1, -2)$ is a point in the fourth. No two points on a graph share the same coordinates unless they are at exactly the same position.

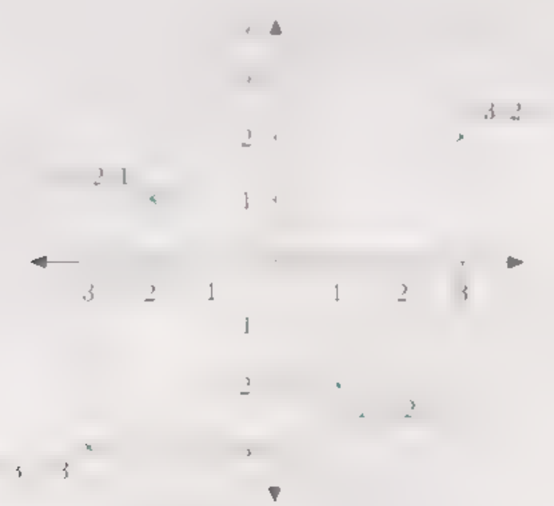


*Figure 18* Plotting coordinate pairs in different quadrants

Gradients for mathematical graphs are calculated in the usual way by dividing vertical distance by horizontal distance. Figure 19 shows a straight-line graph with two points at the coordinates $(1, 1)$ and $(3, 4)$.

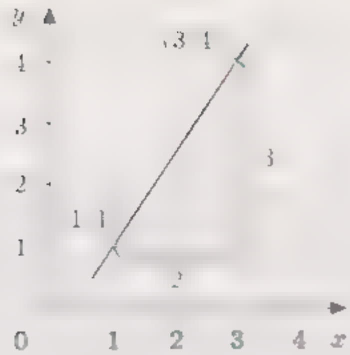



*Figure 19* A straight-line graph with a positive gradient

The vertical distance between the points is just the numerical difference between the two $y$-coordinates, which is $4 - 1 = 3$. The horizontal distance is the difference between the two $x$-coordinates, which is $3 - 1 = 2$. As the $x$-coordinate increases (from 1 to 3), the corresponding $y$-coordinate also increases (from 1 to 4):

$$\text{gradient} = \frac{\text{increase in } y\text{-coordinate}}{\text{increase in } x\text{-coordinate}} = \frac{3}{2} = 1.5$$

Since an increase in the $x$-coordinate is matched by an increase in the $y$-coordinate, the graph has a positive gradient. A straight-line graph with a positive gradient always slopes upwards from left to right.

Now look at Figure 20. Here, the plotted coordinates are (1, 4) and (3, 1). The vertical distance is still 3 and the horizontal distance is still 2, but the graph is now sloping down from left to right rather than up. This time, as the $x$-coordinate increases (from 1 to 3), the $y$-coordinate decreases (from 4 to 1).

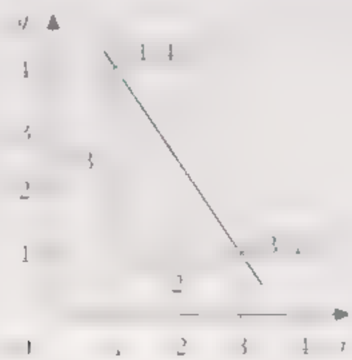


*Figure 20* A straight-line graph with a negative gradient

Treating the change in the $y$-coordinate as a negative increase (as with bank accounts: a reduction in savings can be thought of as a negative increase in savings), write the change as a negative number:

$$\text{gradient} = \frac{\text{increase in } y\text{-coordinate}}{\text{increase in } x\text{-coordinate}} = \frac{-3}{2} = -1.5$$

Recall you did this in *Unit 6* to work out slopes where the height of the ground falls as distance increases.

Sometimes you may come across the words 'positive' and 'negative' written as '+ve' and '−ve', as in '+ve increase' or '−ve slope'.

The graph has a negative gradient. A straight-line graph with a negative gradient always slopes downwards from left to right.

### *Negative gradients*

In all the real-world examples so far, the gradient in the relationship formula has been positive. This is because in each case of conversions, increasing one quantity has resulted in an increase in the other (imagine getting *fewer* francs for *more pounds*!). There are situations, however, where *increasing* one quantity results in a decrease in the other, and conversely. In these cases, the constant in the formula is negative, as is the gradient of the corresponding straight line.

## *Activity 10* *Graph gradients*

and $C = (5, 4)$.

Plot these points on a graph and calculate the gradients of the lines joining $A$ to $B$, and $B$ to $C$.

The values of the $x$- and $y$-coordinates in a graph sometimes relate to *measurements* of physical quantities: for example, in graphs of height against distance, or temperature against time. Physical quantities always have units associated with them, and these must be shown on the axes' labels of the graph.

In mathematics, however, values of $x$- and $y$-coordinates that have been calculated using a formula may simply be numbers: they may not have units attached to them.

Here is an example of just such a relationship. For each value on the $x$-axis the corresponding value on the $y$-axis is given by the formula:

$$\text{value of the } y\text{-coordinate} = (\text{value of the } x\text{-coordinate})^3 \qquad (4)$$

Recall that the cube of $x$ is just the value of $x$ multiplied by itself and by itself again.

This is said as "that the value of the $y$-coordinate is equal to the cube of the value of the $x$-coordinate".

So, for example, when the value of the $x$-coordinate is $-3$, the value of the corresponding $y$-coordinate is $(-3)^3 = -3 \times -3 \times -3 = -27$. When the value of the $x$-coordinate is 0.4, the corresponding value of the $y$-coordinate is $(0.4)^3 = 0.4 \times 0.4 \times 0.4 = 0.064$, and so on.

By convention, the value of each $y$-coordinate is said to *depend* on the value of the associated $x$-coordinate. That is, choose the value of an $x$-coordinate and then use the mathematical relationship to work out the

value of the corresponding $y$-coordinate. The values of the $x$- and $y$-coordinates are referred to as *variables*, because their values are not single fixed numbers. Mathematicians sometimes call the $x$-coordinate the *independent* variable and the corresponding $y$-coordinate the *dependent* variable. But do not confuse dependence with physical causality between the associated quantities.

### Activity 11 Plotting a cubic relationship

Complete Table 2 for the relationship (4) on page 28. The $y$-coordinate is equal to the cube of the $x$-coordinate. Use your calculator to display the data as a line graph. What would be a suitable display window?

Without doing any calculations, describe briefly how the gradient of the graph changes as the $x$-coordinate changes from $-3$ to $+3$.

*Table 2* Data for the cubic relationship

| Value of $x$-coordinate | Value of $y$-coordinate | Coordinate pair |
|---|---|---|
| $-3$ | $-27$ | $(-3, -27)$ |
| $-2$ | $-8$ | |
| $-1$ | | |
| 0 | | |
| 1 | | |
| 2 | | |
| 3 | | |

## 1.4 What story does this picture tell?

> As to the propriety and justness of representing sums of money, and time, by parts of space, tho' very readily agreed to by most men, yet a few seem to apprehend that there may possibly be some deception in it, of which they are not aware ...
>
> (William Playfair (1786) *The Commercial and Political Atlas*, London)

The political economist William Playfair, who developed many of the graphical representations familiar today, was well aware of the visual impact of graphical presentations and of the impressions they can create. In his book *The Commercial and Political Atlas*, published in London in 1786, Playfair published the critical graphic shown in Figure 21. Writing on Playfair's contribution to graphic design, Edward Tufte commented, 'Accompanied by [Playfair's] polemic against the "ruinous folly" of the British government policy of financing its colonial wars through debt, this graphic is surely the first skyrocketing government debt chart, beginning

Recall there was a historical note on William Playfair in *Unit 1*.

(Tufte, E (1983) *The Visual Display of Quantitative Information*, Graphics Press, Connecticut, p. 65)

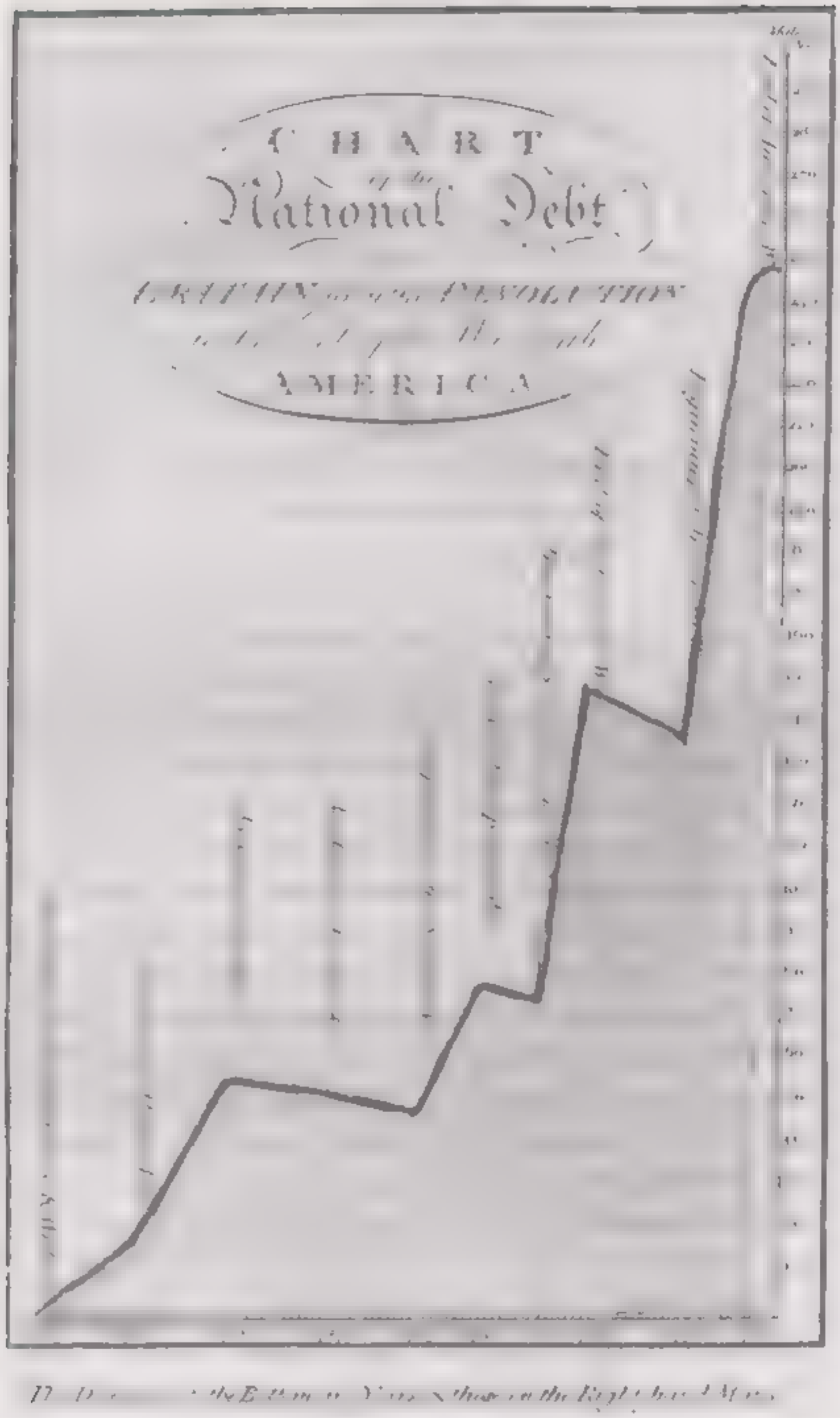



*Figure 21* Skyrocketing debt: from Playfair's *The Commercial and Political Atlas*

the now 200-year history of such displays'. The way Playfair has drawn the graph, using a tall and narrow shape and by not adjusting the money figures for inflation, emphasizes the rapid growth of the British national debt during the eighteenth century.

But Playfair also produced an alternative version a few pages later. Shown in Figure 22, the graph shows the interest on the national debt plotted against time.

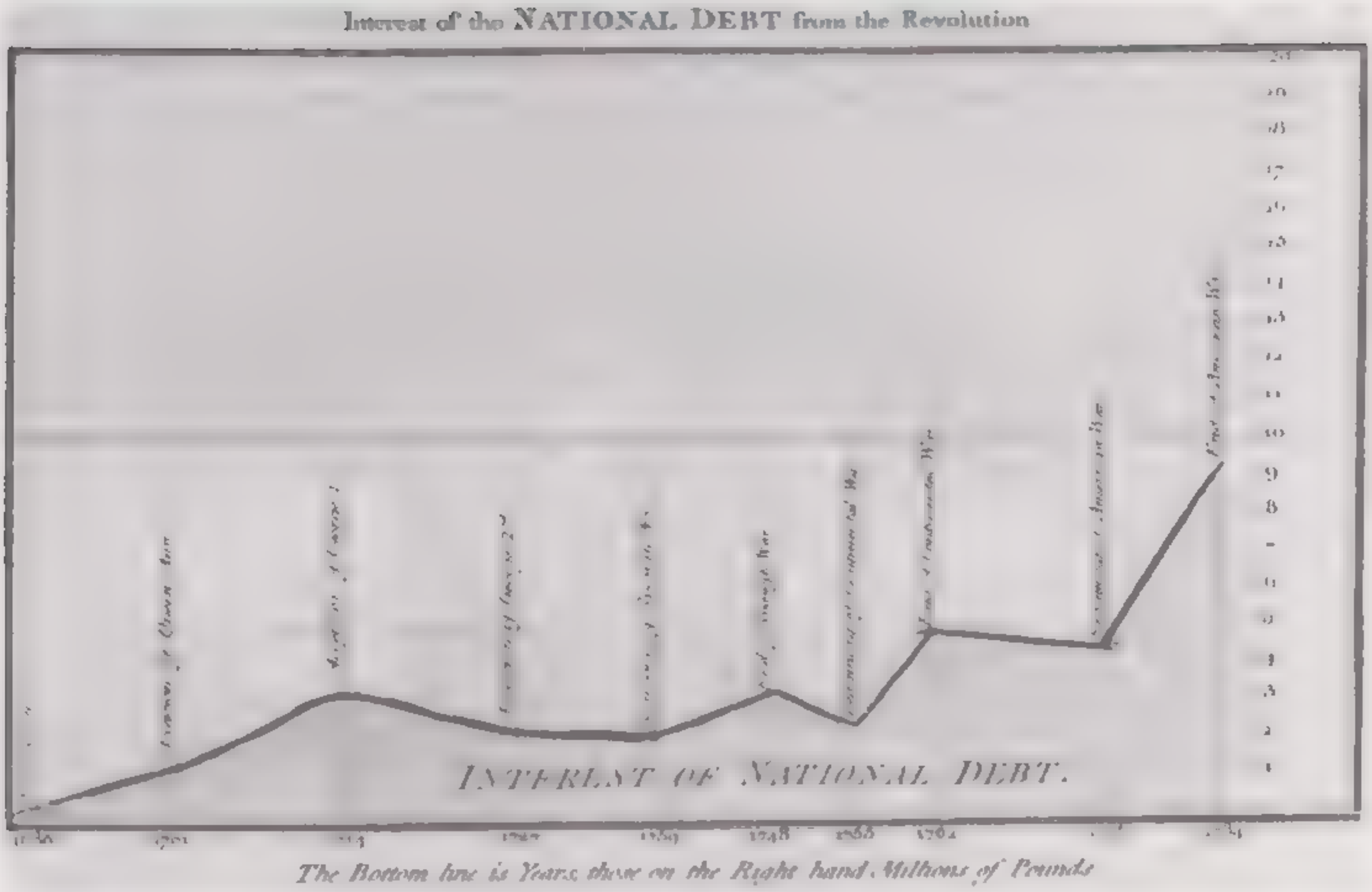



*Figure 22* Interest on the national debt, also from Playfair

Playfair has here taken inflation into account and plotted the cost of the debt in 'real terms'. To lessen the impact further, he has chosen a different format for the graph: a broad horizontal scale for the time and a relatively short vertical scale for the debt. Now the situation does not look quite so bad, although if you look carefully the graphs show that the debt and the interest on it increased by about five times from 1739 to 1784: a period during which Britain was involved in wars with Spain, France, and America.

But authors of graphs do not always make their embedded conventions explicit, preferring to rely on the immediate visual impact of the graphic to encourage readers to skip over the details and jump to conclusions. Here is an example. A graph used by a political party on one of a series of publicity postcards, entitled 'A Better Health Service', is shown in Figure 23. It shows how the number of nurses and midwives in the UK rose from 440 000 to 500 000 between 1978 and 1987, while the number of doctors and dentists rose from 81 000 to 95 000.

## *Activity 12* Skyrocketing growth?

What visual impression does the graph in Figure 23 give about the rise in numbers of nurses and midwives, and doctors and dentists over the period in question? What message do you think the graph is being used to convey?

Now look at Figure 23 more carefully and try to read out the actual state of affairs. What methods have been used to create an impression of 'skyrocketing' numbers of medical staff? How might you redraw the graph to modify the visual effect?


*Figure 23* A better health service?

Representing 'sums of money, and time, by parts of space', as Playfair put it, may indeed seem obvious and readily agreed, but nevertheless graphics showing the rise and fall of profits, expenditure or interest rates over time often need to be approached carefully. As the inventor of the bar chart (or bar graph), Playfair might well have raised a quizzical eyebrow at the example in Figure 24 taken from a national newspaper.

This particular bar chart claims to show how the interest rate on fixed rate loans (often taken out for purchasing or improving a house) has been changing. Look at it for a moment. For what purpose do you think was the graph drawn? How has that purpose been achieved?

The height of each vertical bar represents the interest rate in each period. At first glance, it looks as though the right-hand bar is about twice as high as the left-hand one, giving the impression that the rate had doubled. However, if you look at the vertical scale on the left-hand side of the picture scale, you will see that it starts not at zero but at 5%.

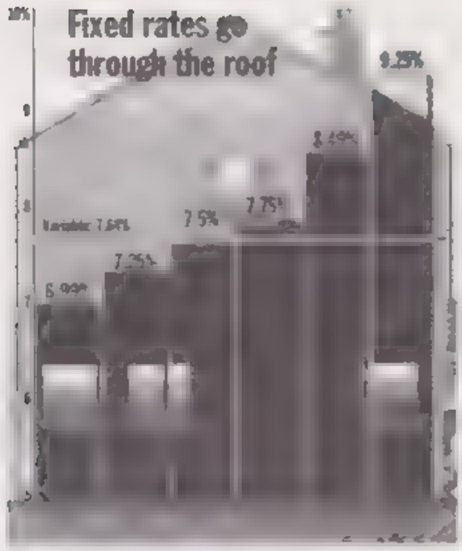



*Figure 24* Fixed rate goes through the roof (source: Halifax)

The purpose of the graph seems to be to show that the interest rate in 1994 rose quite substantially between January and May. This effect has been achieved by carefully selecting the range of values shown on the vertical axis. Although this graph is technically correct, it can give a misleading impression if it is not read carefully. If the graph is redrawn as in Figure 25, so that the vertical axis starts at zero, it gives a rather less dramatic impression of the way in which the interest rate changed.

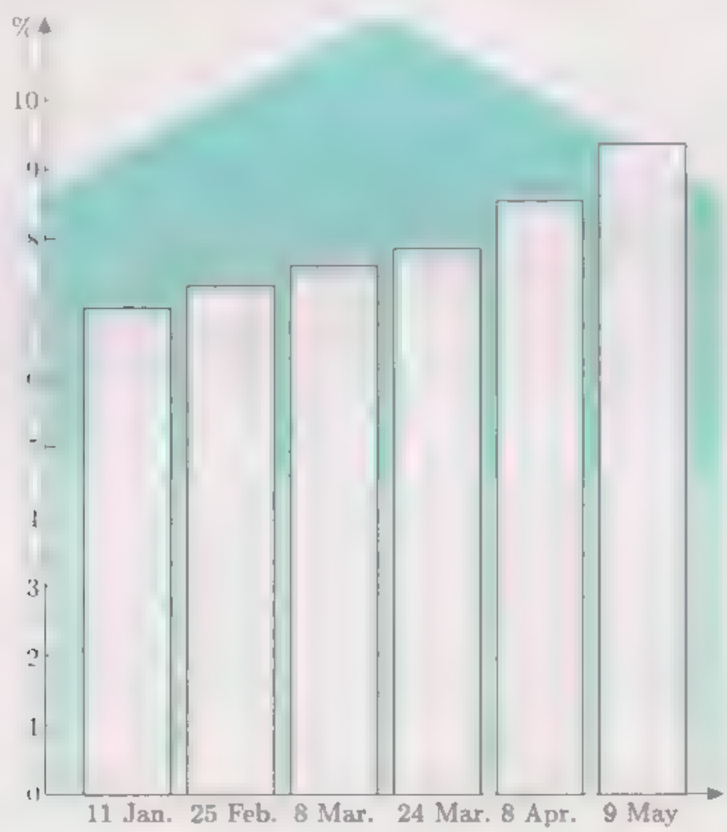



*Figure 25* Fixed rate graph redrawn

Time-series graphs are popular with newspapers for suggesting and comparing trends. But showing how a single quantity varies with time is not the same as showing how two quantities vary, and then suggesting a link between them.

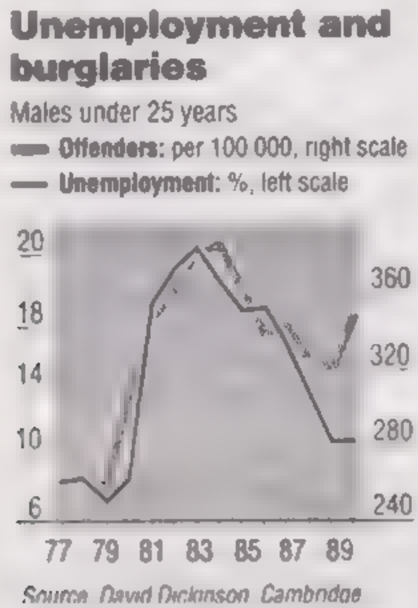



*Figure 26* Unemployment and burglaries

You may recognize this graph from 'Looking with graphs and diagrams' in Section 1 of *Unit 1*. These issues were also touched on in *Units 2* and *3*.

Graphs showing the variation of two things with time often use two different vertical scales. You saw an example of this in the graph in Figure 23 charting the number of medical staff. Figure 26 shows an example taken from a national newspaper. This graph was included in a front-page article suggesting that there is a link between the level of unemployment among young men and the number of offenders committing burglaries. The way the graph has been drawn seems unambiguously to support the claim that when unemployment rises so does crime and, by virtue of the closeness of the shape of the two curves, carries the strong implication that indeed unemployment *causes* crime.

You are not, of course, expected to draw the alternative conclusion that an increase in crime causes an increase in unemployment!

But, as with the discussion in *Unit 4* on causation in the context of health, do not jump to conclusions. First, look carefully at what the graph shows and read out the information that is actually there. Along the bottom, the scale represents the years 1977 to 1990. The vertical axis on the left-hand side shows the level of unemployment among men under twenty-five years old expressed as a percentage. Notice that the scale divisions are 4%, except for the top one which is 2%, although this may be a misprint and '20' should have been printed as '22'.

On the right-hand side, the scale shows the number of offenders per 100 000. Note that the graph on its own does not make it clear just what this scale means. Is it the number of offenders per 100 000 men under twenty-five, or might it be the number of offenders per 100 000 *unemployed* men under twenty-five? The graph gives no clues, so you would have to look elsewhere for clarification, emphasizing the point that all graphs are part of a wider context. Now look at the line graphs themselves. There are

two lines, one relating to the left-hand scale and one relating to the right-hand scale. The two vertical scales have been chosen so that both graphs occupy roughly the same vertical height and, if you look at the bottom left of the graph, start together. The conclusion, of course, is that as unemployment goes up, so does crime, with the further implication being that it is the unemployed who turn to crime. To make that conclusion you are asked to compare trends, but detailed comparison is difficult because the vertical axis of each graph is different. The graphic encourages you to think is that there is a strong causal link between two different trends, by the visual impression created by the way it has been drawn.

But you can play the same game. Figure 27 shows the same data, only now the vertical axes are both scaled in percentages. The left-hand axis still shows the percentage level of unemployment but now the right-hand axis shows the number of offenders expressed as a percentage. This time you could argue that the graph tells quite a different story—that the level of crime is hardly affected at all by the level of unemployment. In spite of a significant 13% (or is it 17%?) increase in joblessness between 1979 and 1983, the number of offenders increased by less than 0.2 percent.

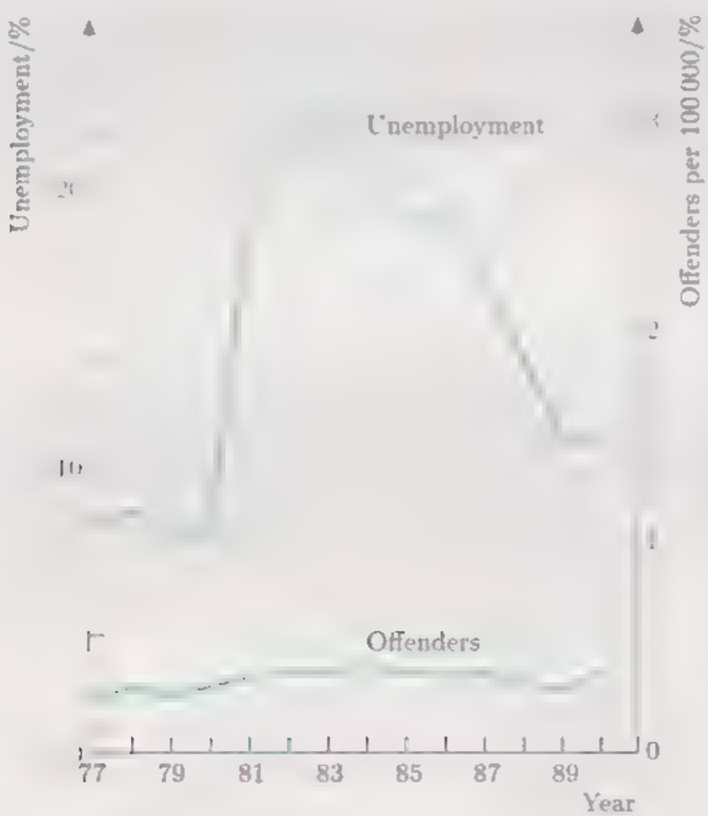



*Figure 27* A different perspective

Even the original graphic starts to tell a different story towards the end of the 1980s, revealing that in 1990 the level of unemployment had dropped back almost to the 10% level of 1980, while crime was not far below its 1984 peak. The strong visual impression of the two overlaid graphs and

the apparent close match between 1977 and 1983 works to divert attention from what is going on in the last years of the decade. What the graphic actually shows is no more and no less than two separate time-series graphs that have been drawn in the same place. There may or may not be a causal link between crime and unemployment, but graphical similarity on its own does not tell about cause. For that you need additional knowledge about the factors and forces that influence an actual, real-world situation.

Comparing trends requires a notion that the variables plotted against time are somehow related. But any such relationship must be established elsewhere—the graph itself cannot do it. A graph is a presentational device, all it can do is display data in a chosen format. Graphs are drawn by people, and it is people who decide what a graph shows and how it shows it. There is nothing inevitable about a graph.

Graphical representations are not restricted to two or three variables. The story a graphic tells can be dramatically enhanced by allowing additional variables to play a subtle counterpoint to the main theme. Perhaps one of the most famous multivariate narrative graphics, using six plotted variables, is the one by Minard examined in the next activity.

## Activity 13 *The fate of Napoleon's army*

Study the reader article 'Narrative graphics of space and time' by Edward Tufte. Use Minard's graphic showing the fate of Napoleon's army to tackle these questions.

(a) When did the army reach Smolensk during the retreat from Moscow?

(b) What was the size of the army at that stage, as a percentage of the size at the beginning of the campaign?

(c) The winter was bitterly cold. Minard's graphic gives the temperature in degrees on the Reaumur scale (which we shall write as °R). On this scale, water freezes at 0 and boils at 80 °R. What was the lowest temperature in °C that the army had to endure?

(d) How many of the men who were at Moscow made it back to the River Niemen?

In summary, this section has looked at time-series graphs, conversion graphs and mathematical graphs. Like all representations, graphs draw from a range of common conventions and styles to convey meaning. From a mathematical point of view, graphs give a visual impression of the relationship between two (or sometimes more) variables, but bear in mind that this impression is largely under the control of whoever draws the graph. When you are drawing graphs for yourself or others, you need to choose and indicate axis labels and scales with care. When you are reading and interpreting a graph, you need to be clear about the context in which the graph exists, and to think about what decisions have led to the graph looking the way it does.

## *Activity 14* Summarizing graphs

Graphs are a very important tool in mathematics, and one which you will meet many times in the course, so make some notes about graphs and the different uses you have come across.

Go back through Section 1 and make notes in your handbook about the different uses for graphs. You might want to include the main characteristics of each graph, and perhaps give an example. Continue making handbook notes as you work through the rest of this unit.

You have seen that there are conventions for drawing graphs, such as scaling and labelling axes, including information about units [illegible] appropriate, giving a title, and so on. Make a list of graph-dr[illegible] conventions, and add to this list as you come across others. Make sure that any new terms introduced in the unit are included in your Handbook sheet.

There is a printed response sheet for this activ[illegible]

## *Outcomes*

After studying this section, you should be able to:

- explain in English and by using examples, the conventions and language used in graph drawing to someone not taking the course (Activity 14);
- use the following terms accurately, and be able to explain them to someone else: 'time-series graph', 'conversion graph', 'directly proportional relationship',' "straight-line" relationship', 'gradient', 'intercept', '$x$-coordinate', '$y$-coordinate', 'coordinate pair', 'variable', 'independent variable, dependent variable (Activities 2, 8, 9, 10, 14);
- draw a graph on a sheet of graph paper, from a table of data, correctly plotting the points, labelling the graph and scaling and labelling the axes (Activities 10 and 11);
- draw and use a graph to convert between a quantity measured in one system of units to the same quantity measured in a different system (Activities 4, 5, 6, 7);
- write down the formula of a straight-line graph, and be able to explain, using sketches, the meaning of the terms 'gradient' and 'intercept' (Activities 8 and 9);
- comment critically on a graph by carefully reading out information (Activities 3, 12, 13).

# 2 Modelling a journey

*Aims* The main aim of this section is to introduce the distance–time graph as a mathematical model of a journey. ◇

You will need graph paper for this section.

Like any mathematical model, a distance–time graph stresses some features of the situation it claims to represent and ignores others. Bear this in mind as you work through this section, and note for yourself which aspects of a journey are described graphically and which do not feature in the model.

## 2.1 Distance, speed and time

▶ Which mathematical formulas are used to relate distance, speed and time?

Look first at distance. If you are travelling at a speed of 30 kilometres per hour, in one hour you will cover a distance of $30 \times 1 = 30$ kilometres. At a speed of 40 kilometres per hour, in 2 hours you will cover a distance of $40 \times 2 = 80$ kilometres. And at 50 kilometres per hour, in 3 hours you will cover a distance of $50 \times 3 = 150$ kilometres.

▶ How is distance related to speed and time in general?

The word formula is 'distance is equal to speed multiplied by time' or, using some symbols:

$$\text{distance} = \text{speed} \times \text{time}$$

Recall the audiotape discussion 'Communicating mathematically', from the preparatory package.

Instead of writing the words out in full each time, you can use a shorthand to speed things up. In mathematics, single letters are often used to stand for quantities described in words or phrases. A common convention is to use the first letter of the word as a way of remembering what quantity it stands for. In this case, therefore, $d$ can be used to stand for the numerical distance, $s$ can be used to stand for the value of the speed and $t$ can be used to stand for the measure of time. So the formula is written like this:

$$d = s \times t$$

This formula is a mathematical model of distance, expressed in terms of speed and time. So if you know what speed you will be travelling at and how long you will be travelling for, you can use the formula to predict how far you will go instead of having to make the actual journey. But the model contains an important assumption.

▶ Can you see what it is?

The model uses the assumption that the speed is constant over the entire journey. But that is clearly unrealistic: during the course of a journey you slow down, stop and speed up again many times. Your actual speed is not constant but continually changing. More useful, however, is *average* speed.

'Average', here, refers to the mean.

▶ If you travel at an *average* speed of 55 kilometres per hour (sometimes faster, sometimes slower), then after 1.5 hours how far will you have travelled?

Re-interpreting $s$ as the average value of the speed, the formula predicts

$$\begin{aligned} d = s \times t &= 55 \text{ kilometres per hour} \times 1.5 \text{ hours} \\ &= 82.5 \text{ kilometres} \end{aligned}$$

So now you can calculate the distance if you know the average speed and the total time the journey takes. But you can also think of the relationship between distance, average speed and time in another way. Suppose you know the distance between two places and the time it takes to travel between them. How would you calculate the average speed? What formula would you use then?

When you are trying to understand how one quantity is related mathematically to another, it is useful to try one or two calculations with numbers to get a feel for the relationship. Suppose you travelled 40 kilometres and took an hour to do it. What would be your average speed? It would be 40 kilometres an hour. Now suppose that you travelled 60 kilometres in 2 hours. What would be your average speed then? It would be $60/2 = 30$ kilometres per hour.

▶ So what is the word formula relating the average speed to the distance and the travel time?

The average speed is calculated by dividing the distance travelled by the journey time:

$$\text{average speed} = \frac{\text{distance travelled}}{\text{total time}}$$

Using the same letters as before the formula is written like this:

$$s = \frac{d}{t}$$

Finally, how would you work out the time a journey should take if you knew the distance and the average speed? How long, for example, would it take to travel 90 kilometres at an average speed of 30 kilometres per hour? You would cover the distance in $90/30 = 3$ hours.

▶ So what is the general word formula relating travel time to the distance and the average speed?

The travel time is equal to the distance travelled divided by the average speed:

$$\text{total travel time} = \frac{\text{distance travelled}}{\text{average speed}}$$

which you can write as:

$$t = \frac{d}{s}$$

So now you have all three forms of the relationship between time, average speed and distance. Here they are again:

$$d = s \times t, \quad s = \frac{d}{t} \quad \text{and} \quad t = \frac{d}{s}$$

Notice that the first formula is the product of time and speed, while the other two have distance divided by time or speed.

## *Remembering the formulas*

Figure 28 shows a way to remember the formulas. Draw a circle with a '$Y$' in it. Starting at the top of the '$Y$' just write the letters $d$, $s$ and $t$ in the spaces in alphabetical order. (It does not matter which way you go round!) Then to find the formula for time simply cover up 'time' on the diagram, and you are left with distance over speed. Similarly covering up 'speed' gives the formula distance over time. And finally covering up 'distance' gives the formula speed multiplied by time.

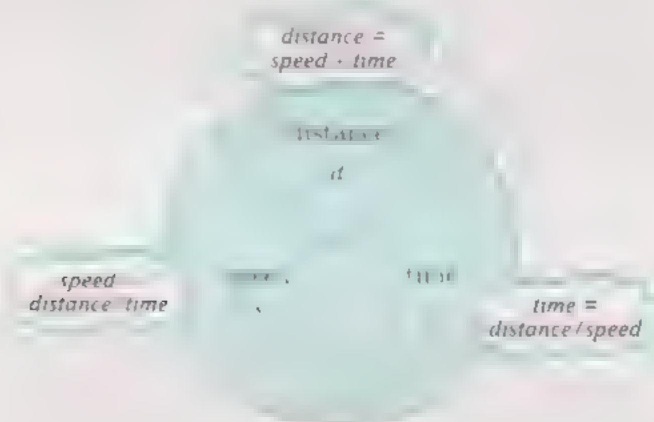



*Figure 28* Remembering the relationship between distance, speed and time

The above is a good example of a *mnemonic*, that is a device intended to help you remember something. It may or may not actually help. Only you can tell. Recall from *Unit 3* the discussion of the W-shaped diagram as a device for picturing the median and quartiles of a batch of data. There are some similarities. But realize that there is nothing mathematical to understand about such memory devices. They have been designed to produce the correct result, but there is no conceptual link between Ys in circles and relationships between speed, distance and time. Mnemonics are about remembering, not understanding.

## Activity 15 *Finding the right formula*

Use the appropriate formula to work out:

(a) the average speed in kilometres per hour, if a distance of 25 km is covered in 45 minutes;

(b) the distance travelled in kilometres after travelling for 30 minutes at an average speed of 75 kilometres per hour;

(c) the time in seconds to cover a distance of 500 metres at an average speed of 10 metres per second.

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The formulas for speed, distance and time are all examples of mathematical models. In fact, you have already met a very similar model in Naismith's rule in *Unit 6*. Here, as then, you should bear in mind that such models stress some aspects of travelling but ignore others. Building a mathematical model involves making some assumptions, and usually this involves disregarding those inconvenient aspects of real-world events which can not easily be fitted into a mathematical description.

Take, for example, the model $s = d/t$ used to calculate speed. Dividing a journey distance by the travelling time gives a single number which represents the average speed on the journey. The formula contains no information about the style of transport, about the joys, delights, delays and frustrations of travelling, about stops for petrol or children being sick. The typical complexities of even an everyday journey have been boiled down to just two numbers—the overall distance and the total time taken.

Recall too the observation 'the map is not the territory' from *Unit 6*. A mathematical model of a journey is not the journey itself.

The relationship between distance, speed and time can serve as the basis for representing a journey as a graph. Recall that it is:

distance = average speed × time

You should recognize this formula as a directly proportional relationship. The constant of proportionality in the relationship is equal to the average speed. At any particular average speed, the distance travelled is directly proportional to the time the journey takes.

▶ If you plot a graph of distance travelled against time, what sort of graph will you get?

Look at Figure 29. The vertical axis represents distance travelled along the route and the horizontal axis represents time. Both are measured from the start of the journey. For any particular average speed, the graph of distance against time is a straight line starting at the origin. The average speed is represented by the gradient, or slope, of the graph. This type of graph is called a ***distance–time graph***.

A graph like this is described mathematically by the following 'straight-line' relationship.

value on vertical axis = gradient × value on horizontal axis

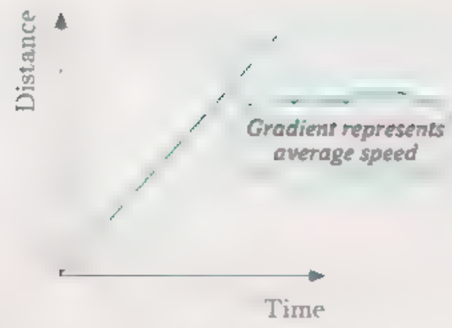


*Figure 29* Distance–time graph

## *Example 1* London to Paris

The Eurostar train service that connects London and Paris via the tunnel under the English Channel (*la Manche*) covers a distance of about 380 km in three hours in 1996. Assuming a constant speed, what would the distance–time graph of this journey look like?

Take the *Gare du Nord* (the Northern Station) in Paris as the start and measure time and distance from there. The vertical axis on Figure 30 represents distance, in kilometres, from Paris along the path of the railway track, and the horizontal axis represents the elapsed time, in hours after leaving Paris. The origin of the distance–time graph, the point $(0, 0)$, represents the starting point of the journey, and the point $(3, 380)$ represents the arrival of the train at Waterloo Station in London some 380 km away and 3 hours later. The straight line connecting the points represents the journey from Paris to London, and the gradient of the graph represents the average speed of the journey. In this case, it is $380/3 = 127$ km per hour.

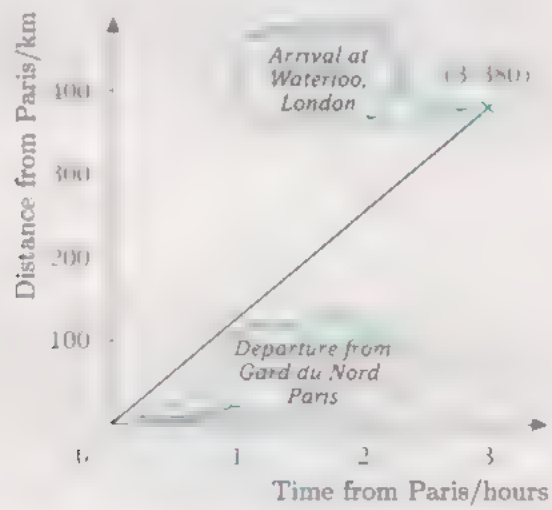


*Figure 30* Distance–time graph of the Paris-London journey

Figures 31 and 32 show graphically the effect on the journey time of changing the average speed. Increasing the average speed, as in Figure 31, increases the slope of the graph and the journey time is shortened. Reducing the average speed, as in Figure 32, reduces the slope of the graph and lengthens the journey time.

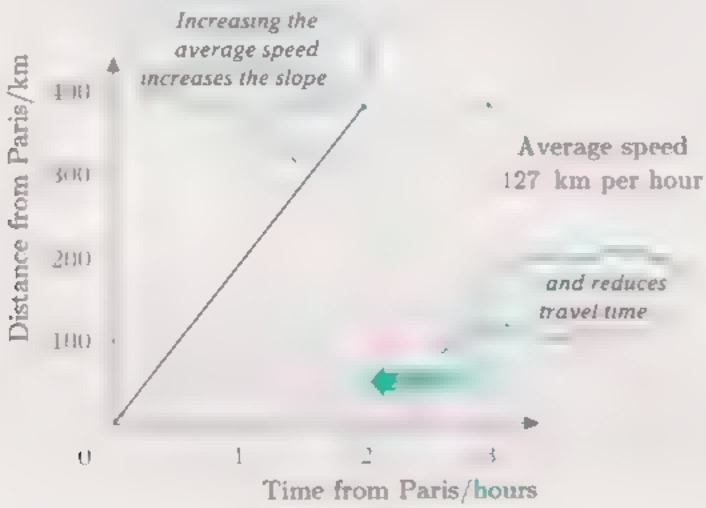


*Figure 31* Increasing the average speed

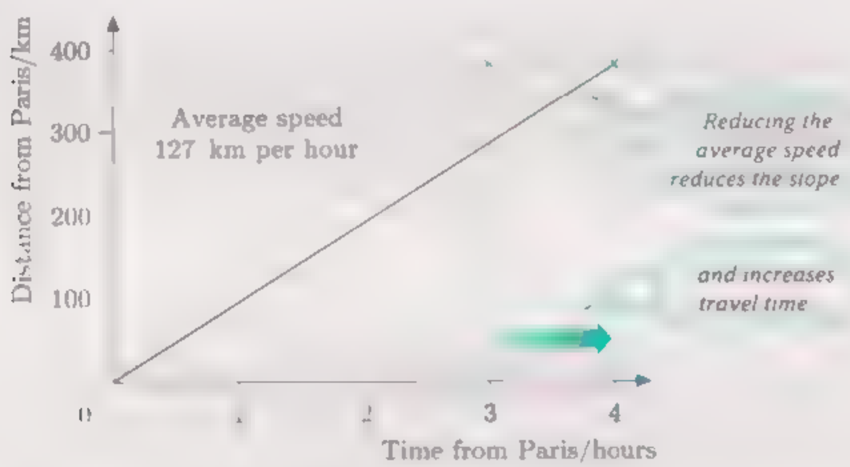


*Figure 32* Reducing the average speed

The distance–time graphs in Figures 30, 31 and 32 are based on average speed. But the train does not travel at a constant speed throughout the journey: it will travel faster on some sections of the line than on others.

▶ How can this be shown on the distance–time graph?

First, look at a possible journey in more detail to set up a graphical model of the distance–time relationship. To get started, split the journey into three sections: the journey from Paris to the tunnel, the journey through the tunnel, and the journey from the tunnel to London. An initial assumption is that the train travels at a constant, but different, speed over

each section. This simplistic model ignores the details of an actual journey (such as stops at stations and local speed restrictions) to concentrate on the more general features of the distance–time graph.

The first section is from Paris to the tunnel entrance near Calais. This distance is roughly 230 km, so if the train travels at an average speed of 300 km per hour it will take about $230/300 = 0.77$ hours, or around 46 minutes. This part of the journey is represented by the distance–time graph in Figure 33.

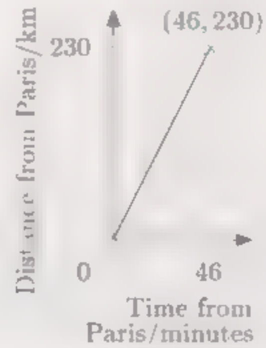


*Figure 33* The Paris–tunnel distance–time graph

Now for the 50 km journey through the tunnel. The train's average speed drops to about 160 km per hour over this stretch, so the travel time through the tunnel is about $50/160 = 0.31$ hours, or just under 20 minutes. This section of the journey begins where the previous section finished — at Calais. So represent it on the distance–time graph as in Figure 34 by drawing another straight line starting where the previous graph ended.

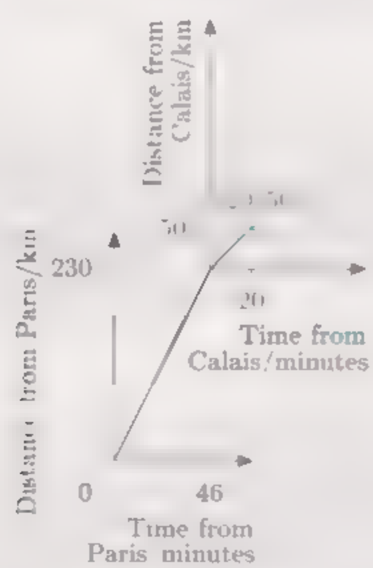


*Figure 34* Adding the tunnel section to the distance–time graph

The final stage of the journey is from the tunnel exit near Folkestone to Waterloo in London, a journey of about 100 km. This part of the journey takes about 114 minutes, or 1.9 hours (making a journey time of 180 minutes or 3 hours overall), so the average speed is $100/1.9 = 53$ km per hour. Once again, as Figure 35 shows, the distance–time graph is extended by joining on the graph representing this final section.

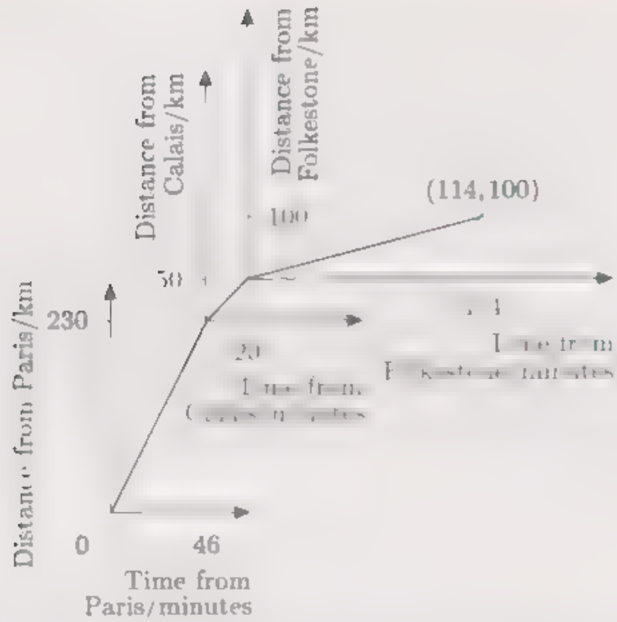


*Figure 35* Adding the final tunnel–London section to the distance–time graph

The three separate lines are combined into one overall distance–time graph representing the entire journey, as shown in Figure 36. The times for the sections are added together, so that the scale on the horizontal axis shows the total time that has elapsed since leaving Paris. Similarly the distances of the sections are combined, so that the scale on the vertical axis shows the total distance from Paris. On the graph, the point (46, 230) represents the ending of the journey across northern France and the start of the journey through the tunnel, and the point (66, 280) represents the emergence of the train from the tunnel and the start of the final part of the journey into London.

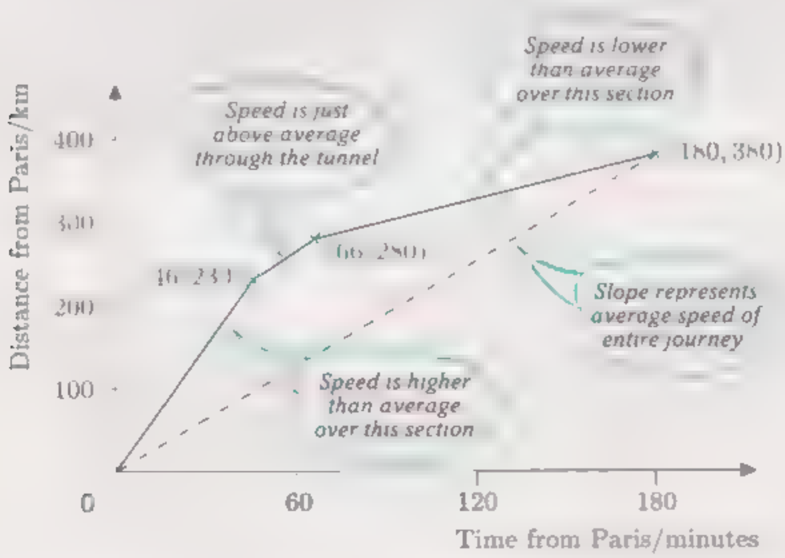


*Figure 36* The overall Paris–London distance–time graph

The first thing to notice about the overall distance–time graph is that it is not one straight line so it cannot be represented by a simple proportional relationship, although it was built up by looking at the proportional distance–time relationships for each section. There are three straight-line sections to the graph. Look at the slopes. The slope or gradient of the first section, representing the journey from Paris to the tunnel, is the steepest of the three, indicating that the train travels at its highest speed across northern France. The train slows down for its 50 km journey through the tunnel, its lower average speed represented by the shallower slope of the central straight-line section of the graph. Emerging from the tunnel, the train slows down further for its journey across southeast England into London. This final section of the graph has the smallest gradient (slowest speed) of all. In each of the three sections of Figure 36, the slope of the graph differs from the average slope. This means that the average speed of the train for each of these three parts differs from its overall average speed. From Paris to the tunnel the speed is considerably greater than the average. Through the tunnel, the speed is slightly more than the average, while through south-east England the speed is less than the average.

## Activity 16 *Interpreting a distance–time graph*

Rana delivers newspapers. Figure 37 is a distance–time graph of her round.

Using the graph, answer the following questions.

(a) On which section of the journey do you think she was walking the fastest? Make some brief notes to explain your answer.

(b) What is your interpretation of section CD?

(c) Part of her round is up a hill where her walking speed is slowest. Which section of the graph do you think represents this?

(d) Which part of the graph represents Rana's return to her starting place. Why?


*Figure 37* Distance–time graph of Rana's round

Distance–time graphs are a means of replacing a description given in words by a mathematical description of the same event. What follows is a narrative account: that is, a description in the form of story about a bicycle ride. Read the story and then think about how you would use this account to produce a mathematical model of the ride in the form of a distance–time graph.

> Sunday started a bit cloudy. The temperature was about 13 °C, but I thought I'd keep to the original plan and go cycling with the kids on the track around the local reservoir. The eight-year-old has got his own bike; I can hire a bike for me with the four-year-old on a seat on the back when we get there. What we usually do is to cycle from the bike-hire place to a pub where we can sit outside and have some lunch—burger and chips probably. The pub's not far—about 5 km from the start and it takes us about 25 minutes to get there. We usually stop for about 45 minutes. After lunch, we go on another 2 kilometres, stop for about 15 minutes and then head back the way we came. I guess our average cycling speed between stops is about 10 km per hour. On the way back, we usually stop at a playground for half an hour. From there it is 3 km to the bike-hire shop which takes us about 15 minutes.

First of all, notice that this narrative account contains a mix of information. There is speed and time and distance information to be sure but there are also other items which you will not be able to fit easily into a mathematical description, such as the comments about the weather, the location of the cycleway, the ages of the children, and what was for lunch. All these details are important from a personal point of view; but as far as the mathematical model of the bicycle ride is concerned they have no bearing whatsoever. Mathematics, therefore, is not an alternative language: it has no means of speaking of many events that people find important. What it does offer in this example, however, is a way of revealing and representing very specific features of the journey that are embedded in the narrative account.

To build a mathematical model of the bicycle ride you need to be selective about the information you choose. You may also have to piece information together and make some assumptions. Narrative accounts are not mathematical accounts and there may be some inconsistencies you have to resolve before you can put together a reasonable mathematical story.

## Activity 17 *A mathematical story*

Use the information in the story and the relationships between distance, average speed and time to complete Table 3, given overleaf.

*Table 3* Time and distances for the cycle ride

| Time from start/minutes | Distance from start/km (along the route) | Comments |
|---|---|---|
| 0 | 0 | Leave bike-hire |
| 25 | 5 | Reach pub |
| 70 | 5 | Leave pub |
| | | Short break |
| | | Start back |
| | | Playground stop |
| | | Leave playground |
| | | Arrive at bike-hire |

A partially-drawn distance–time graph of the cycle ride is shown in Figure 38. The first straight-line section represents the outward 5 km journey to the pub, which takes 25 minutes. The average speed is represented by the gradient of the graph, which is:

$$\frac{\text{distance travelled}}{\text{total time}} = \frac{5}{25} = 0.2\,\text{km per minute.}$$

This is equivalent to $0.2 \times 60 = 12$ km per hour.


*Figure 38* Part of the graph of the cycle ride

During the lunch stop, the distance from the start does not change. Obviously the speed is zero, so the graph is a straight line with a slope of zero. The length of the line represents the length of time (45 minutes) spent at the pub. After lunch, the journey continues for 2 km at an average speed of 10 km per hour. This part of the journey therefore takes $2 \div 10 = 0.2$ hours (remember that travel time is equal to distance travelled divided by average speed), or 12 minutes. After this short ride comes the 15-minute break. Once again, the distance from the start does not change over this time, and so this section of the distance–time graph is a horizontal line too.

## Activity 18 Completing the graph

Complete the distance–time graph in Figure 38 for the journey back to the cycle hire shop. What is the average speed for this part of the ride?

You should now be able to interpret distance–time graphs, and be able to use them to find information about the average speed, the distance travelled and the time taken for different sections of a journey. Given any two of these quantities you should be able to identify and use the appropriate formula to find the third.

An important feature of a straight-line graph is its gradient. The gradient, or slope, of a graph expresses a relationship between a change measured along the horizontal axis and the corresponding change measured along the vertical axis. The steepness of the slope indicates how fast the variable represented on the vertical axis is changing with respect to the variable on the horizontal axis. So a steep slope represents a rapid *rate of change* of one variable with respect to the other, and a more gentle slope represents a lower rate of change. On a distance–time graph, the gradient is the rate at which distance is changing with respect to time. In other words, the rate of change of distance with time is a measure of speed.

Recall in *Unit 6* that the gradient of a sloping hillside related the change in height to the change in horizontal distance.

## Activity 19 Impossible journey?

Figure 39 shows three distance–time graphs. For each graph, explain using brief notes, whether or not it represents a possible journey.

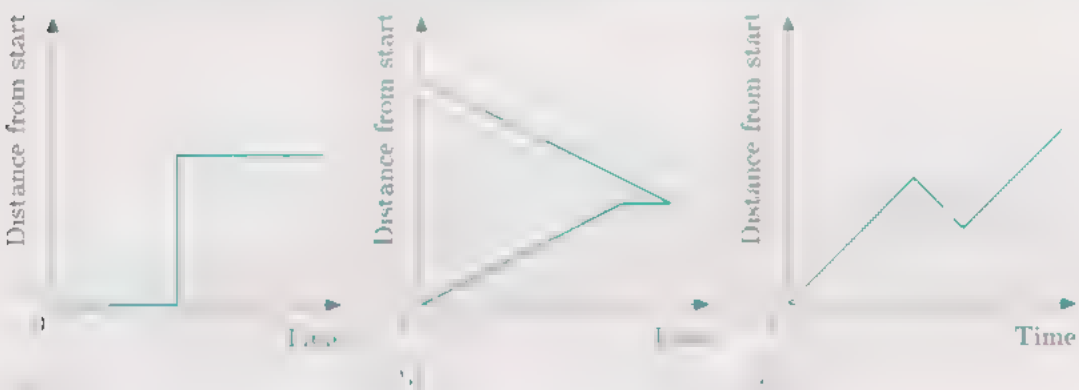


*Figure 39* What stories do these distance–time graphs tell?

## 2.2 A mathematician's journey

Distance–time graphs can show more than one journey on the same graph. The journeys do not have to start from the same place, or start at the same time, but all times and distances must be measured from a common origin along a common route. In this subsection, you will see how drawing a distance–time graph can help in planning a journey.

Bob and Alice both work for the Open University. Bob lives in Edinburgh in Scotland and Alice lives in Milton Keynes in England about 510 kilometres to the south. During a phone call they discover that Alice will be travelling 340 km north to the Newcastle Regional Centre and Bob will be travelling 420 km south to an OU summer school in Nottingham on the same day.

You met network maps in *Unit 6*. A network map ignores all features of a journey except place names and distances.

Figure 40 shows a network map of their journeys, indicating the distances between the cities. They both aim to start their journeys at about 10 am.

It is unlikely that Alice would actually drive through Nottingham on her way to Newcastle. However, Nottingham lies very close to the motorway she would be using. For this model, ignore this detail, and assume Nottingham is directly on Alice's route.


*Figure 40* Network map of Alice's and Bob's journeys

Mathematical models can be used to describe events that have occurred, or to predict how events will go in the future.

Will they pass each other on the road going in opposite directions, and if so, can they arrange to stop and meet at a convenient point? Alice—a mathematician—offers to draw up a distance-time graph to model the journey. Alice will use her model to *predict* how the actual journeys might go. To get started, she needs to make some assumptions. She can build these into her model and then check to see whether the predictions it gives seem reasonable.

She estimates she will drive for about 2 hours at an average speed of about 85 kilometres per hour and then stop for a break for about 30 minutes. She then intends to continue her journey to arrive finally in Newcastle around 3.30 pm. Alice also needs Bob's estimates. He reckons to drive for about $2\frac{1}{2}$ hours to cover 170 kilometres, stop for about 30 minutes and then press on for Nottingham to get there at about 5 pm.

Alice summarizes their planned journeys in two tables. She measures times from 10 am, and distances from Milton Keynes. Table 4 shows Alice's journey times (in hours) from her 10 am starting time, and the distance (in kilometres) she has travelled from Milton Keynes. You can see that at the start of the journey she will be 0 km from home, and 2 hours later she will be $2 \times 85 = 170$ kilometres away. The distance does not change while she has her break. After her break, the remaining 170 km to Newcastle should take about 3 hours at an average speed of $170/3 = 56$ km per hour.

*Table 4* Alice's journey

| [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
|---|---|---|---|---|
| Time after start (hours) | 0 | 2 | 2.5 | 5.5 |
| Alice's distance from Milton Keynes (kilometres) | 0 | $2 \times 85 = 170$ | $170 + 0 = 170$ | $170 + (3 \times 56) = 340$ |

Now look at Table 5 showing Bob's predicted journey. He also starts at 10 am and aims to cover 170 km in about $2\frac{1}{2}$ hours, an average speed of 68 km per hour. After a 30-minute stop near Newcastle, he reckons to drive the remaining 250 km to reach Nottingham 4 hours later, at about 5 pm. To achieve this his average speed must be $250/4 = 62.5$ km per hour. Bob will be driving away from Edinburgh and towards Milton Keynes (although his destination is, of course, Nottingham), so his distance from Milton Keynes will decrease with time.

*Table 5* Bob's journey

| [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
|---|---|---|---|---|
| Time after start (hours) | 0 | 2.5 | 3 | 7 |
| Bob's distance from Edinburgh (kilometres) | 0 | $2.5 \times 68 = 170$ | $170 + 0 = 170$ | $170 + (4 \times 62.5) = 420$ |
| Bob's distance from Milton Keynes (kilometres) | 510 | $510 - 170 = 340$ | 340 | $510 - 420 = 90$ |

Figure 41 (overleaf) shows the distance-time graphs for Alice's and Bob's proposed journeys.

The horizontal axis shows the time of day, and the vertical axis shows the distance in kilometres from Alice's starting place in Milton Keynes along the route. Milton Keynes is the reference point for all measurements of distance. For Alice's journey, the first part of the graph is a straight line with a positive slope (the line slopes up from left to right). The gradient of the line represents a steady speed of 85 km per hour. During her break, her speed will be zero and hence the graph is a horizontal line with a gradient of zero. The second part of her journey is modelled by another straight line with a positive slope, this time indicating an average speed of 56 km per hour. The positive slopes indicate that Alice's distance from Milton Keynes will *increase* with time.

Bob's distance–time graph is similar to Alice's except that the slopes of the lines (apart from the break) are negative, they slope down from left to right. The negative slope means that Bob's distance from Milton Keynes will *decrease* with time as he drives south. In other words, the gradient of a line of a distance–time graph contains information about the direction as well as the speed, of travel.

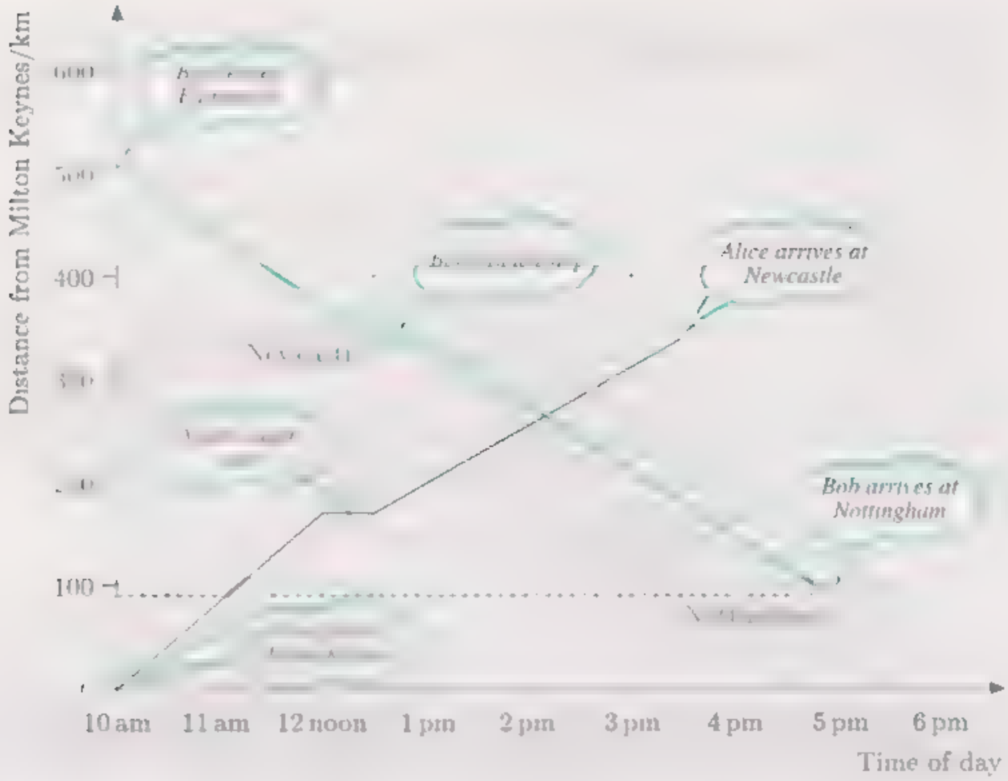


*Figure 41* Distance–time graphs for Alice's and Bob's journeys

The gradient of a distance–time graph can be positive, negative or zero, depending on whether travel is away from, towards, or stationary relative to, the starting point of the journey. Speed itself, however, is always a positive quantity. What the gradient indicates is *velocity*. Velocity is speed in a particular direction. On a distance time graph, a positive gradient represents a positive velocity: that is, a speed in a direction away from the place represented by the origin of the graph. A negative gradient represents a negative velocity, that is speed in a direction towards the origin.

The distance–time graph for Alice's journey in Figure 41 shows positive gradients, indicating positive velocity, because she will be going *away* from Milton Keynes, the reference point for distance measurements. Bob's graph, on the other hand, shows negative gradients, indicating negative velocity because he will be travelling *towards* Milton Keynes. Alice's estimated speed during the first section of her journey is 85 km per hour. Her corresponding velocity, however, will be 85 km per hour *away* from Milton Keynes. A statement of velocity, therefore, must include both the speed and the direction of travel.

The point at which two distance–time graphs cross has a special significance. In this case, each graph represents Alice's or Bob's distance from Milton Keynes as time increases. The point at which the graphs cross represents the situation where Alice and Bob are exactly the same distance from Milton Keynes at the same time. The model predicts that, at that moment, they both will be in exactly the same place, although travelling in opposite directions and on opposite sides of the road.

## Activity 20 *Making predictions*

If Alice and Bob follow their plans:

(a) How far apart will they be at 1.30 pm?

(b) When and where will they pass each other?

By drawing a distance–time graph, Alice has predicted that she and Bob will pass on the stretch of road between Newcastle and Nottingham. Using the OU's computer system, she sends an e-mail message to Bob suggesting that they meet at a roadside restaurant about 275 km north of Milton Keynes (for Bob this will be $510 - 275 = 235$ km south of Edinburgh). Bob acknowledges her e-mail and the meeting is set up.

'e-mail' stands for electronic mail.

Alice guesses they will probably stop for about 30 minutes. But what effect will this have on the times they will reach their respective destinations? She can modify her graphical model to include the stop to predict the consequences.

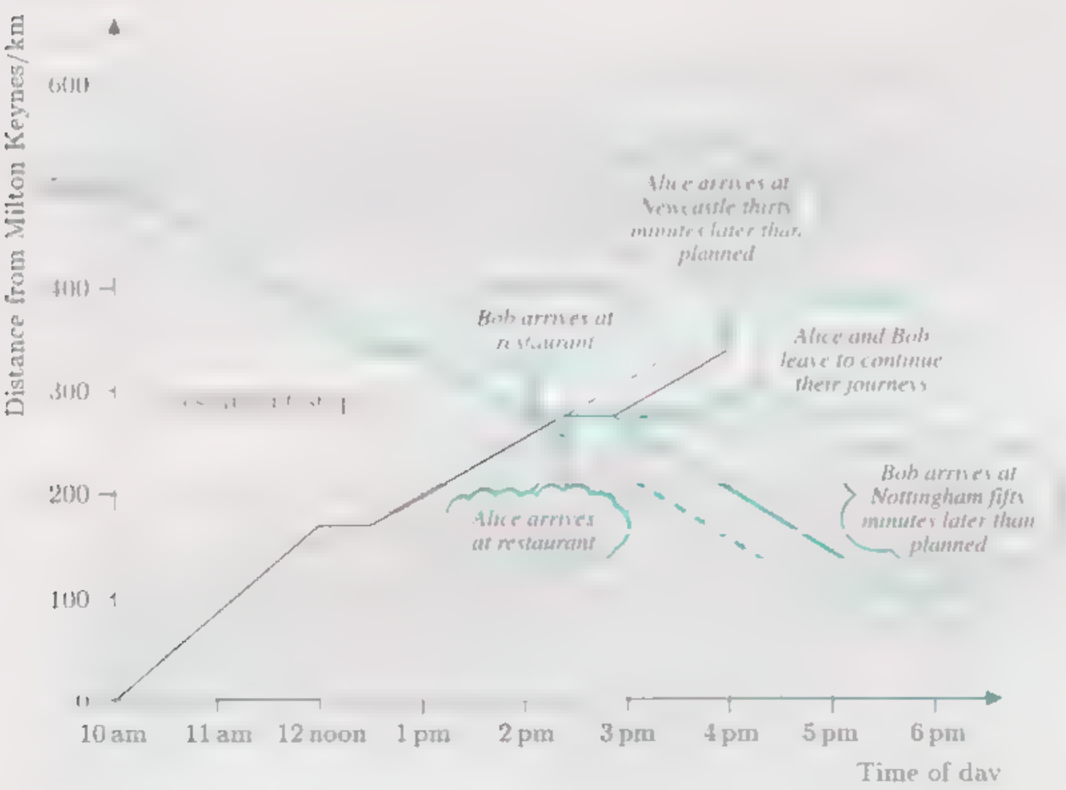


*Figure 42* The modified distance–time graphs

Figure 42 shows the modified distance–time graphs. Bob's graph shows that he will probably arrive at the restaurant first at about 2.00 pm. Alice is likely to arrive about 20 minutes later. If they stay for 30 minutes and both leave around 2.50 pm, Bob will have been there for nearly 50 minutes. So he is likely to complete his journey about 50 minutes later than he originally planned. Since Alice made only a 30-minute stop, she should get to Newcastle at about 4.00 pm, 30 minutes later than she had originally planned.

## *Activity 21* Using the model for planning

a) How would you use Figure 42 to find out what Alice's average speed after lunch should be if she wants to arrive at the restaurant at the same time as Bob? Note down the steps you would take to find the [illegible]

b) What will be the average speeds for Bob and Alice's complete journeys if they keep to their plan?

A distance–time graph is a graph of distance measured from a specific place and along a particular route, plotted against time measured after a specific time. The gradient on such a graph gives the numerical value of the average speed and indicates the direction of travel. Speed is an unsigned quantity equal to the distance travelled along a set route divided by time irrespective of direction towards or away from the starting point (the origin for distance measurement). A positive gradient indicates travel away from the starting point and a negative gradient indicates travel towards the starting point. Speed and direction together give the velocity of travel, which is represented by the gradient of the graph.

## *Activity 22* Planning a journey

Use a series of labelled diagrams to show how you would construct and use a distance–time graph for planning a journey. Your audience is a group of people planning a sponsored cycle ride. You do not know them, but you are aware they are not confident with mathematics.

There is a printed response sheet for this activity.

## *Outcomes*

After studying this section, you should be able to:

- use the following terms accurately and be able to explain them to someone else: 'average speed', 'velocity', 'distance–time graph' and how one could be used to plan a journey (Activity 22);
- explain and use the mathematical relationships between distance, time, average speed and the gradient of a distance–time graph (Activities 15, 21, 22);
- construct a distance–time graph from a narrative account of a journey (Activities 17, 18);
- draw correctly, use and interpret distance–time graphs (Activities 16, 18, 19, 20, 21);

# 3 On the right lines

*Aims* The main aim of this section is to show an application of distance–time graphs in the operation of a railway service. ◇

This section uses the video band 'Single-track blunders' to illustrate how distance–time graphs are drawn and interpreted by the timetable planners of a small railway company, and shows the role of this graphical technique in planning a flexible service. Graphical representations of journeys have been used for over a century by travel organizations to plan timetables. Figure 43 shows a graphic produced by the Frenchman E. J. Marey in 1885 showing the schedule for trains between Paris and Lyon. The stations are listed along the vertical axis and are separated in proportion to their actual distance apart. The horizontal axis shows a time span of 24 hours from 6 am through midday (*mıdi*), midnight (*mınuıt*) until 6 am the following morning. Each sloping line represents a scheduled train, with the slope of the line representing the train's average speed. The faster the train, the steeper the line is. The horizontal lines indicate the time the train is stopped at a station.

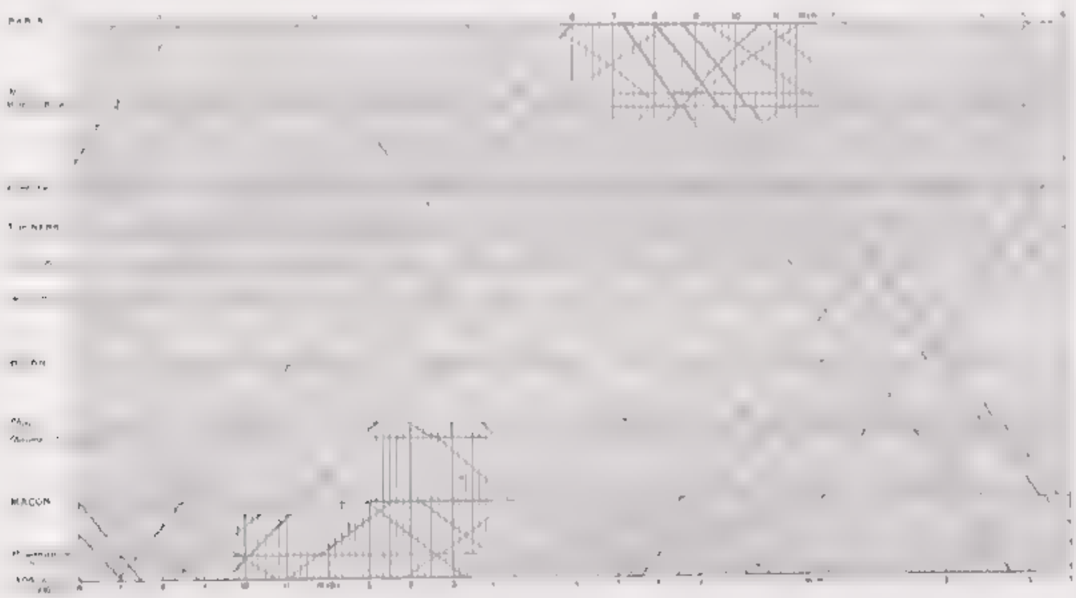


*Figure 43* Marey's travel graph (Source: E. J. Marey, *La Méthode Graphique*)

Trains travelling away from Paris are indicated by lines with negative slopes, and those travelling towards Paris by lines with positive slopes. The intersection of two lines indicates that two trains travelling in opposite directions are at the same place at the same time; these are the locations at which trains pass each other.

Marey's graph shows clearly how important the Paris-Lyon route was, even in the 1880s, with eight trains leaving Paris every day. Even a fast train, however, took over nine hours to make the journey. By comparison Figure 44 shows the path of the modern high-speed TGV (*Train à Grande Vitesse*) service overlaid on the original 1880s schedule. The TGV makes the journey in under three hours. These distance–time graphs make a powerful statement of the continuing national importance of the Paris-Lyon link.

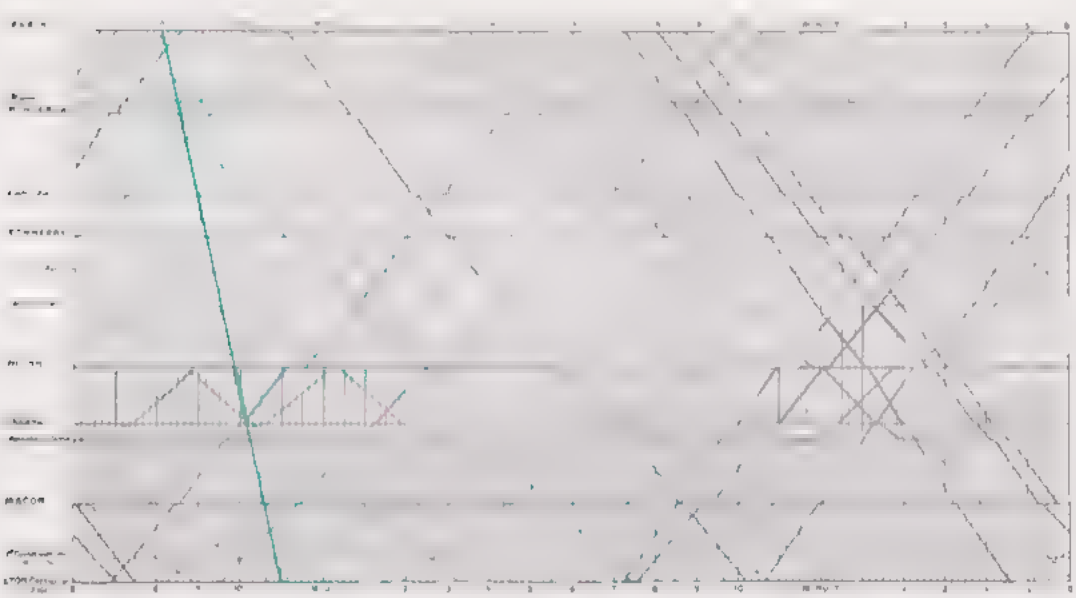

*Figure 44* The TGV service overlaid on the 1880's schedule

Distance–time graphs and timetables used to be drawn up by hand. Now computers do the detailed work associated with scheduling in large-scale public transport systems. However, distance–time graphs still find a place in planning and operating smaller services.

## 3.1 Single-track minders

You should read through this subsection, including the activities at the end, and then watch the video band 'Single-track minders' in the parts indicated by the activities. The video lasts about twenty-five minutes. At certain points you will be asked to stop the tape and complete an activity.

In the 1960s, many of the UK's passenger and freight railway services were closed down as part of an economic re-evaluation of the railway system. Some lines were dismantled and forgotten, but others attracted railway enthusiasts and preservationists determined to re-open lines and run public passenger services using the old steam locomotives and rolling stock.

There are now a number of these smaller railway companies operating in the UK. They provide an attraction for the public while also functioning as

working museums. As a part of the growing leisure and heritage industries these companies cannot operate without skilled staff knowledgeable about railway practices and procedures. But neither can they afford to ignore the financial side of their business.

Although much of the work is done by volunteers, the railway companies must still operate profitably. Since their services are non-essential in the sense that the railways do not usually carry freight or provide regular passenger services for commuters, they must rely on visitors. Visitors are unlikely to keep coming back for the same train ride, however, so the companies must provide a range of different attractions, such as putting on trains for parties or special occasions and running special events for children during school holidays or at Christmas.

For the people running the railway all this adds up to a demand for flexible operations. The railway must be able to vary its services according to the time of year, to schedule extra trains for special events, and to cope with a continuous programme of maintenance and restoration work. All this, as well as ensuring safety for the travelling public and the railway staff themselves.

The video looks at one such small railway – the Severn Valley Railway – which runs almost 25 kilometres (nearly 16 miles) between Kidderminster and Bridgnorth in the West Midlands of England. Most of the journey is along a single-track line. At busy times, two or more trains may be running in opposite directions. To minimize the risk of a head-on collision, therefore, operating [illegible] must be designed carefully.

In the video, you will see how distance–time graphs are used to represent and plan the movement of trains up and down the single-track line. Timetables can then be constructed and checked using these graphs.

Now watch the video band 'Single-track minders'. There are four activities associated with the video sequence. Try the appropriate activity when you are asked to stop the tape. You should tackle Activities 25 and 26 at the end of the band.

Recall the comments from the Study guide for this unit about ways you might work on this video band.

***Now watch band 6 of Videotape 1 until the first tape stop.***

## Activity 23 *Graphing the journey*

On a sheet of graph paper, plot a distance–time graph for the journey from Bridgnorth to Kidderminster. (Note that the graph shown on the video is actually a position–time graph, because actual places rather than distances from a single place are shown on the vertical axis.) The map shown on the video is repeated in Figure 45. Note down briefly what information you will need to plot the graph. (You should ignore the data for the bridge and tunnel given in Figure 45.)

Position–time graphs are explored in more detail in *Unit 11*.

Work out the highest and lowest speeds reached on the journey. In what sections of the line do they occur?

What is the average speed for the complete journey?

Make some notes to explain how you made these calculations.


*Figure 45* A map of the journey from Bridgnorth to Kidderminster from the video

***Restart the video and watch until the next tape stop.***

## *Activity 24* Take the A train

Using the UP and DOWN timetables in Figure 46, draw a distance–time graph for the trains AN1 and AS1. Start the graph for AN1 at Bridgnorth at 10.35 am and finish back at Bridgnorth at 1.18 pm. The graph for AS1 should start at Kidderminster at 10.45 am and finish back at Kidderminster at 1.11 pm.

On its return journey to Bridgnorth, train AN1 leaves Bewdley at 12.58 pm. How long must it wait at Arley if AS1, returning from Bridgnorth, is on time? If the AN1 travelled from Bewdley at 35 km per hour, how long would it have to wait at Arley before it could continue?

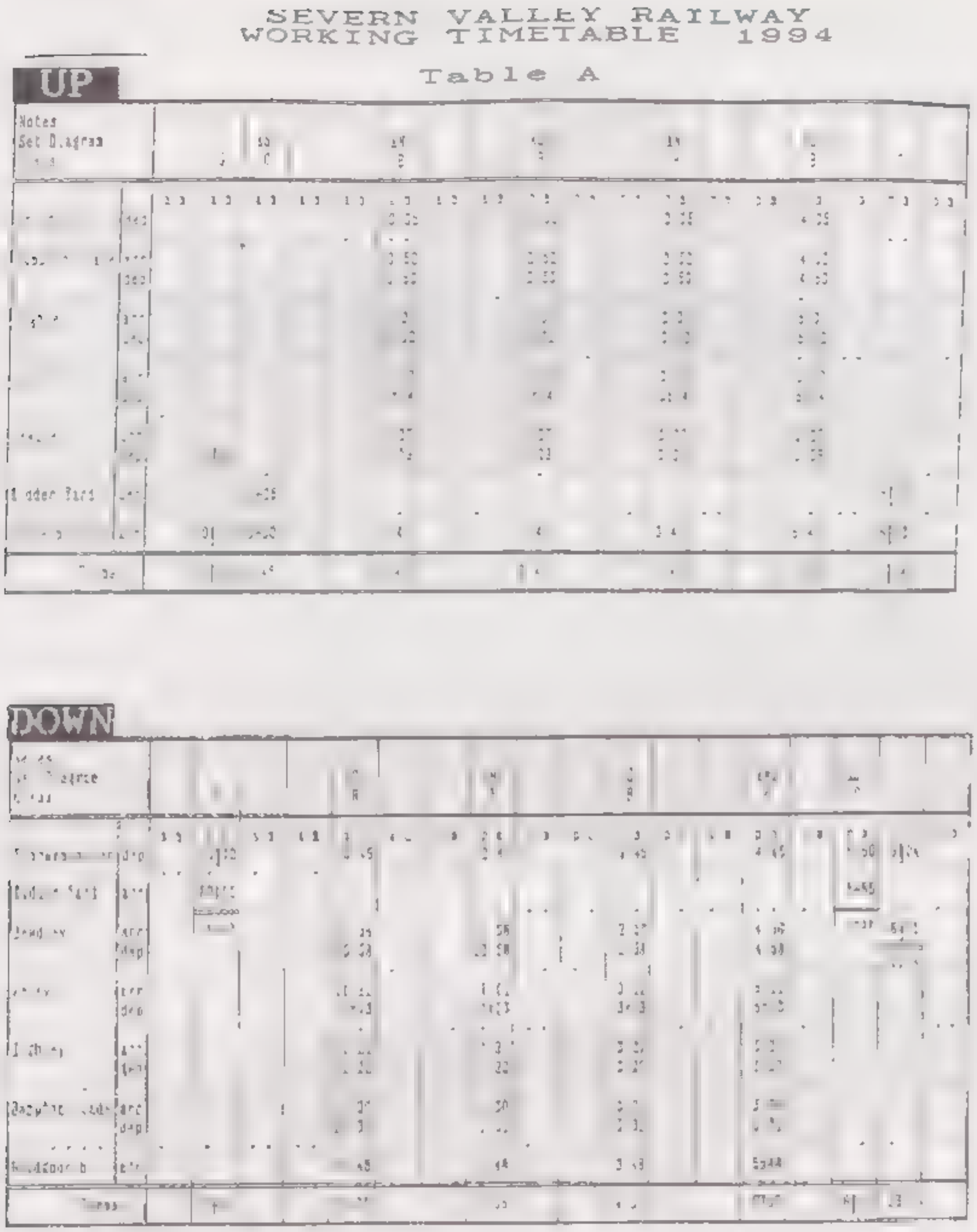


*Figure 46* Severn Valley Railway timetable

***Restart the video once again and watch to the end of the band.***

## *Activity 25* *Planning a special*

A special train has been booked to take a party from Bridgnorth to Kidderminster. Use your distance–time graph to plan an extra train that will leave Bridgnorth at 11.13 am, cross the service train at Hampton

Loade and wait there for 30 minutes. It is then to continue non-stop to Kidderminster, crossing with the service train at Bewdley.

(a) When should the train reach Bewdley?

(b) What should be its average speed between Hampton Loade and Bewdley?

(c) What time should it arrive at Kidderminster, assuming it maintains a constant speed from Hampton Loade?

## Activity 26 *Timetabling a Santa Special*

At Christmas, Santa Special trains run between Kidderminster and Arley. The service runs every thirty minutes, leaving Kidderminster at fifteen and forty-five minutes past the hour.

Figure 47 shows the position–time graphs of two Santa Special trains. Between these trains two more trains run to complete the service.

Draw in the graph for the train that leaves Kidderminster at 11.15 am. It runs at exactly the same speed as the 10.45 and the 11.45 trains. Trains can cross just outside Kidderminster.

(a) When does it arrive at and depart from Bewdley?

(b) At what time will it reach Arley?

Draw in the graph for the train that arrives at Kidderminster at 11.46 am.

(c) When does it arrive at and depart from Bewdley?

(d) At what time does it leave Arley?

(e) Could another service be fitted in so that trains at fifteen-minute intervals?

Make a few notes to explain how you obtained your answer.


*Figure 47* Position–time graph for two Santa Specials

Timetables and distance–time graphs are different representations of scheduled train movements. They are both models which can be used to *predict* when trains will run, to *analyse and compare* different schedules when problems occur, and to *design* new operating schedules to meet new demands. Both models provide information which allows the company to operate safely and flexibly. The information is used by different groups of people:

- by passengers to plan their trips;
- by track staff to be aware of train movements;
- by signalling staff to ensure the safety of trains travelling in opposite directions on the line;
- by planning staff to schedule new or additional services.

## Activity 27

Which model would be more useful to each of the above groups of people?

## Outcomes

After studying this section, you should be able to:

- draw, interpret and use distance–time and position–time graphs in a specific context (Activities 23 to 26);
- record how you tackle mathematical problems (Activities 23–26);
- comment on the usefulness of tables and graphs for different purposes (Activity 27).

# 4 A calculated plot

*Aims* The aim of this section is to develop your skills in using the calculator to display graphs from lists of data, and from formulas. ◇

From your work in Chapter 6 of the *Calculator Book*, you should know how to enter lists of data into your calculator, perform calculations on those lists, and display the results as a graph.

Working with lists is useful when you need to repeat a particular type of calculation over and over again. You would need to do this if, for example, you want to calculate average speeds from a list of time and distance information. But not all graphs and calculations will be based on a list of numerical data. In Subsection 1.2, on conversion graphs, you saw that relationships between quantities are often more conveniently expressed as formulas. The formula of a 'straight line' needs information only about the gradient and the intercept to describe the line completely and uniquely.

In Chapter 7 of the *Calculator Book*, you will see how to enter a formula into your calculator so you can display a graph of the relationship. Moving from a formula to a graph provides a new view of a mathematical relationship, and the facilities of your calculator enable you to explore that relationship quickly and easily.

Chapter 7 also introduces you to a feature of the calculator that you have not used before. This is the programming facility, which allows you to set up short sequences of instructions—programs—that your calculator will perform at the press of a key. In effect, the programming facility provides you with an 'extra key' on your calculator—one that you can set up to perform automatically any sequence of operations that you have specified.

***Work through Chapter 7 in the Calculator Book.***

## Activity 28 *Learning File and Handbook update*

Make sure that you have completed both the Learning File and the Handbook activities for this unit.

First, look back to your response for the Learning File activity (Activity 2) that you started in Section 1. Which different methods have you been using to learn about graphs? Make a list of them in order of usefulness.

You have also seen examples of different types of graphs, and you have seen a specific application of distance–time graphs on the video. Has this been useful to help you think about the pros and cons of displaying information graphically?

Now review the methods you chose to help the group of market researchers. Do you want to amend your list? Make some brief notes to explain your decision.

If you have other points to add to the Handbook sheet, take time to do this now. Make sure you have completed the review sheet.

---

## *Outcomes*

You should now feel more confident in using your calculator to:

- ◇ display graphs based on lists of data
- ◇ perform calculations on lists of data
- ◇ enter the formula of a straight line that corresponds to its associated graph
- ◇ enter a two- or three-line program [illegible]

# *Unit summary and outcomes*

This unit has been about drawing and interpreting graphs used for different purposes.

Section 1 looked at time-series graphs, conversion graphs and mathematical graphs. These are all graphical representations showing how one variable changes relative to another. Time-series graphs show how a measured quantity varies with time, while conversion graphs give a visual representation of the relationship between quantities measured in one system of units and the same quantity measured in another. Many conversions are represented by straight-line graphs, and you saw how these could be represented by a formula involving the gradient and the intercept of the straight line. Mathematical graphs may be more complex than simple straight lines, and may be used to illustrate relationships (usually described by a formula) which do not have a particular physical interpretation.

Section 2 looked at formulas relating distance, speed and time and also at distance–time graphs. You saw how formulas can be concisely expressed in letters rather than words, providing a different symbolic representation of a relationship—one which is looked at in detail in the next unit. Distance–time graphs offer yet another representation. A distance–time graph is a graphical model of a journey and can be interpreted to give information about speed and the direction of travel.

Section 3 showed an application of distance–time graphs in the planning and operation of a small single-track railway. The video and the activities showed how graphs are used to predict, analyse and schedule train movements so that the railway company operates safely, flexibly and profitably.

In Section 4, you used the calculator to display graphs from lists of numbers and from formulas. By entering a formula directly, the calculator can be used to explore the nature of the mathematical relationship itself.

Graphs, like all representations, are not without bias in some form or another. Any graph is drawn from a particular point of view, to accentuate some features rather than others. The appearance of a graph is a result of people making choices, not of some value-free process detached from human interests and concerns.

## Outcomes

Here is a list of the things you should be able to do when you have finished this unit:

- use the following terms accurately, and be able to explain them to someone else: 'time-series graph', 'conversion graph', 'distance–time graph', 'directly proportional relationship', ' "straight-line" relationship', 'gradient, 'intercept', '$x$-coordinate', '$y$-coordinate', 'coordinate pair', 'variable', 'independent variable', 'dependent variable', 'average speed', 'velocity';
- given a table of data, draw a graph on a sheet of graph paper, correctly plotting the points, labelling the graph and scaling and labelling the axes;
- be able to explain in English and by using examples, the conventions and language used in graph drawing to someone not taking the course;
- draw and use a graph to convert between a quantity measured in one system of units to the same quantity measured in a different system;
- write down the formula of a general straight-line graph, and be able to explain, using sketches, the meaning of the terms 'gradient' and 'intercept';
- comment critically on a graph by carefully reading out information;
- draw correctly, use and interpret distance–time graphs;
- explain and use the mathematical relationships between distance, time and average speed;
- construct a distance–time graph from a narrative account of a journey;
- be aware of the use of letters rather than numbers in mathematical formulas;
- explain in plain English to someone not taking the course how a distance–time graph is constructed, and how it can be used to plan a journey;
- explain in your own words how the gradient of a distance–time graph is related to the average speed and direction of travel;
- record how you tackle a mathematical problem in a specific context.

You should also feel more confident in using your calculator to:

- display graphs based on lists of data;
- perform calculations on lists of data;
- enter the formula of a 'straight-line' relationship and display the associated graph;
- enter a two- or three-line program and run it.

# Comments on Activities

## Activity 1

At this stage, you may spend little time on planning your study, or you may do it routinely. What is important is for you to adopt a system that works for you and that you find useful. Studying and learning at a distance is not easy and many students find that a few minutes spent thinking about how and when they work is time well spent. People plan in different ways, but it is generally the case that planning ahead helps you to be more effective in what you do.

Part of planning is also thinking about your own progress: self-monitoring. How are you getting on? Do you feel you need to spend more time on particular aspects of your studies? Are you going to complete the assignment questions for a unit in the time you have allocated to study the unit? Monitoring your own performance in particular areas helps to give you an insight into your progress generally. This is an important skill to acquire for independent learning. It helps you to focus on those areas you want to improve, and so enables you to communicate more effectively with your tutor.

## Activity 2

You have already done some work relating to graphs on this course and you may well have other experiences of using graphs. You may be used to drawing them, or reading them. This activity asks you to focus on what you consider might be helpful for people who feel they have little understanding of graphs, and who feel they need to be more confident about using them in their work. It may be helpful to think about methods you have found useful, and those that have not been particularly useful in your own learning about graphs. As you work through the unit, make some notes about your approach to working with graphs.

## Activity 3

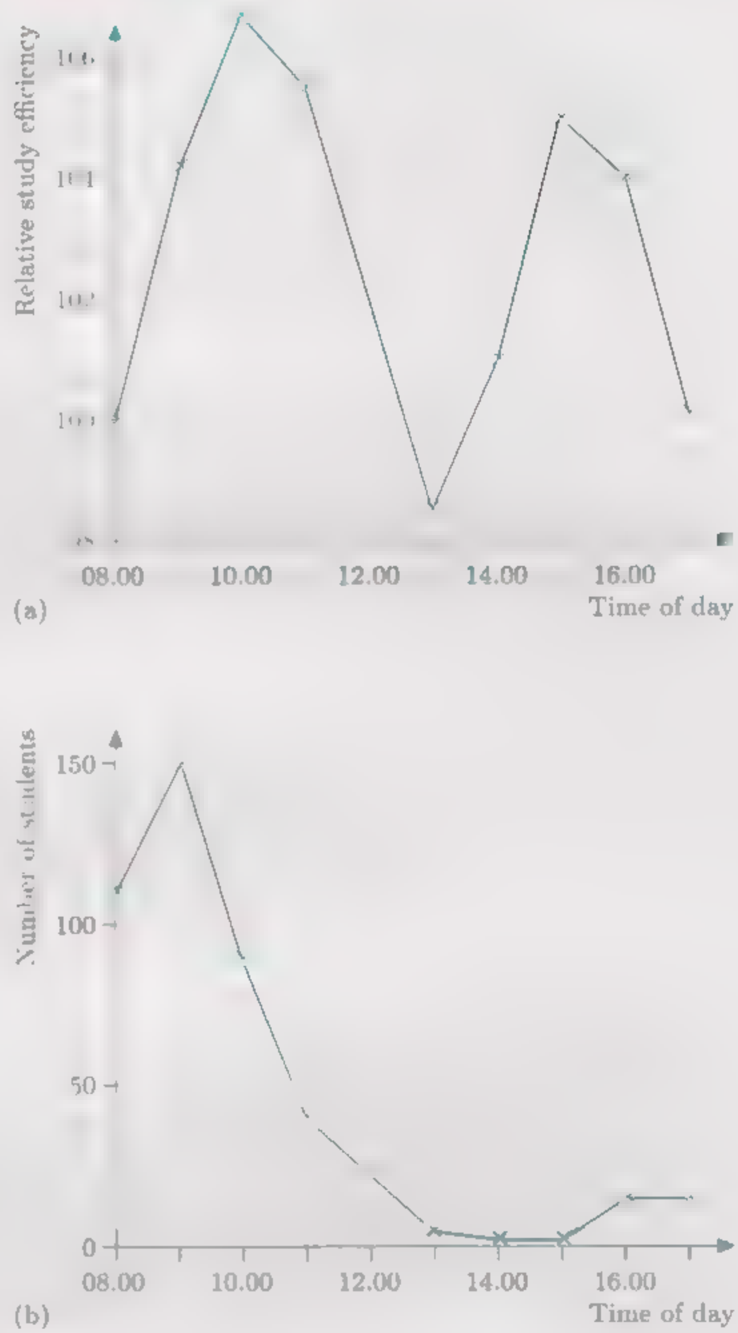


*Figure 48*

Figure 48(a) shows the study efficiency data and Figure 48(b) shows the preferred study time, with each plotted against the time of day (using the 24-hour clock).

Students in the study preferred to work between 8 am and 10 am. The time-series graph shows

that efficiency dropped markedly in the early afternoon. This dip also appears in other studies and seems to be independent of whether or not the subjects had a midday meal. The points on the graphs are connected by straight lines (they also could be connected by a smoothly curving line), indicating that no sudden changes are expected between the data points. From your own experience is this a reasonable assumption?

The graphs suggest that the preferred hours of work do not reflect the times of greatest ability (in terms of study efficiency), since the afternoon is unused. This may mean that students were unaware of the potential of the afternoons for studying, or simply that they preferred to do something else. What are your study patterns and preferences?

## Activity 4

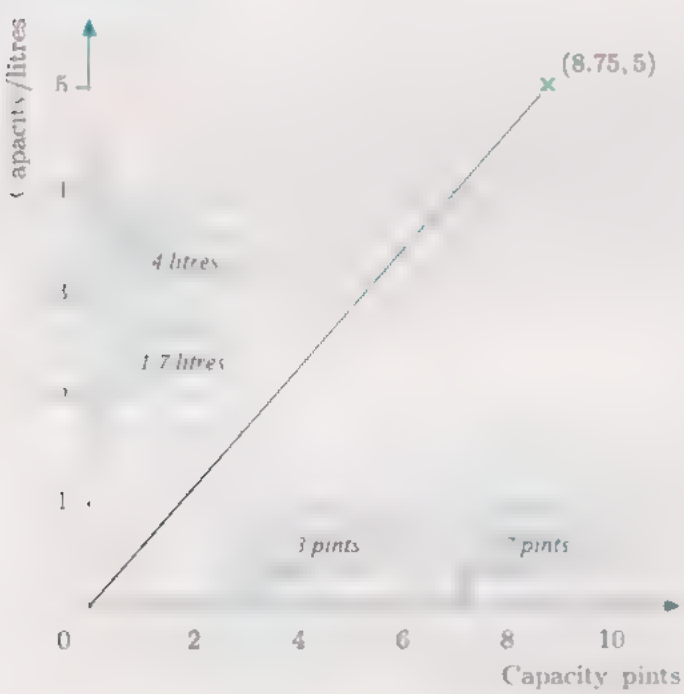


*Figure 49* Graph to convert between pints and litres

Figure 49 shows a graph to convert between pints and litres. 1 litre is about 1.75 pints (actually, a litre is closer to 1.76 pints, but stick with the rhyme!), so to be able to convert 4 litres, the scale along the horizontal axis of the graph needs to extend to at least $4 \times 1.75 = 7$. The graph has been drawn slightly bigger than the minimum size with the horizontal scale representing 0 to 10 pints and the vertical scale representing 0 to 5 litres. The graph was constructed by drawing a straight line from the origin $(0, 0)$ to the point $(8.75, 5)$, representing the equivalence between 5 litres and 8.75 pints. You may have chosen different point and/or scale. Make sure that your graph has a title, and that the axes are scaled and labelled clearly and correctly.

From the graph, 3 pints is equivalent to just over 1.7 litres, and 4 litres is equivalent to 7 pints.

8 pints is equivalent to about 4.6 litres. So the conversion relationship is 1 gallon = 4.6 litres.

Here are some ideas about the pros and cons of tables and graphs. Tables are easy to use and hold a lot of information in a compact way. But they do not give any visual impression of the general mathematical form of the relationship in the way that graphs do. In principle, graphs drawn from formulas can be used to convert between any two equivalent values, provided that the values lie within the range of the graph. Tables give only selected pairs of values, intermediate values have to be estimated. You may have thought of some other points.

## Activity 5

One pound is equivalent to 0.454 kg. So the conversion graph will go through the points $(0, 0)$ and $(1, 0.454)$, as in Figure 50(a). The gradient is

$$\frac{\text{change along vertical axis}}{\text{change along horizontal axis}} = \frac{0.454 - 0}{1 - 0} = 0.454$$

A distance of 1 km is equivalent to 0.621 miles. So the conversion graph will pass through the points $(0, 0)$ and $(0.621, 1)$, as in Figure 50(b). The gradient is

$$\frac{1 - 0}{0.621 - 0} = \frac{1}{0.621} = 1.61$$

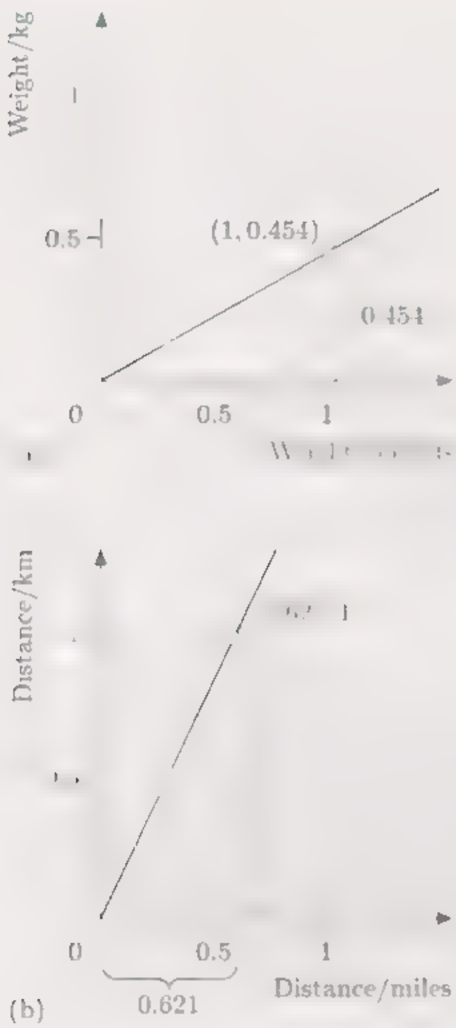


*Figure 50* Graphs to convert between (a) pounds and kilograms, and (b) kilometres and miles

## Activity 6

15 °C is equivalent to 59 °F.

200 °F corresponds to about 90 °C.

## Activity 7

Figure 51 shows the conversion graph extended to cover the range −50 °C to 50 °C. The corresponding range on the Fahrenheit scale is −58 °F to 122 °F.

A temperature of −40 °C corresponds to −40 °F. This is the only point over the entire range at which a single temperature reading on the two scales are numerically equal.

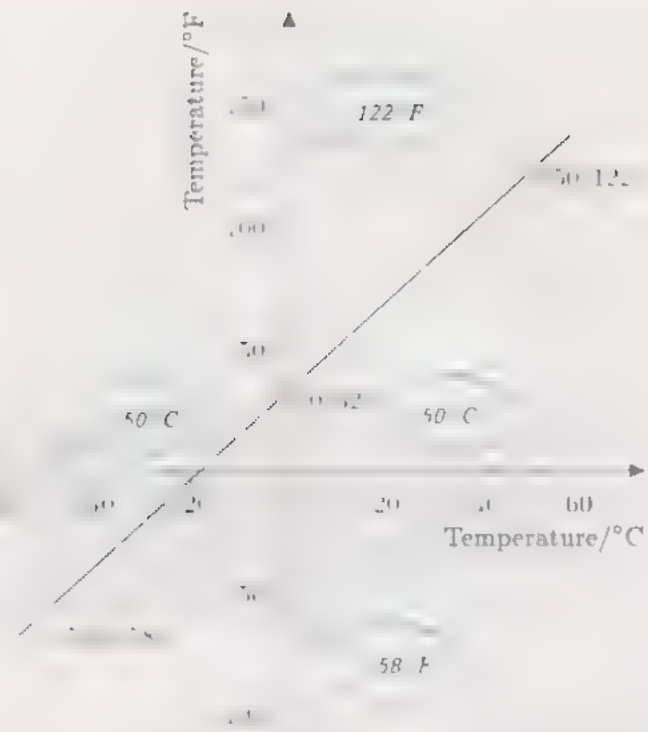


*Figure 51* Extended temperature conversion graph

## Activity 8

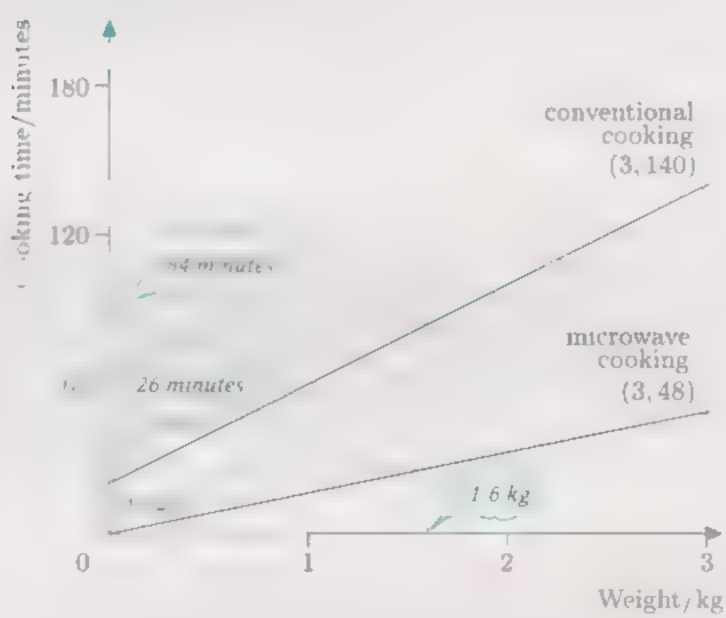


*Figure 52*

The general formula for the cooking time is

time = (time per kg × weight in kg) + extra time

Allowing 40 minutes per kilogram, plus an extra 20 minutes, gives the formula:

cooking time (minutes) = 40 × weight in kg + 20

This is the formula for a straight-line graph with a gradient of 40 and an intercept of 20. Figure 52 shows the relationship. Now check that the graph goes through the right points. When the weight is zero, the line must cross the vertical axis at 20, the value of the intercept. So one point on the graph is $(0, 20)$. Using the cooking time formula you can find another point. A 3 kg chicken will take $(40 \times 3) + 20 = 140$ minutes to cook. So $(3, 140)$ is another point on the graph.

The relationship is really a rule of thumb that has been found to work reasonably well for small whole chickens up to about 3 kg. Common sense should prevail however, and the rule may not be so useful for very small pieces, where the cooking time may be too long. By itself, the mathematics cannot make decisions for you about what is and is not reasonable. Putting a chicken of zero weight into the oven and cooking it for 20 minutes does not make a lot of sense! The same could be said if you extend the rule too far in the other direction, and try to work out the cooking times for excessively large pieces of poultry.

For microwave cooking at 16 minutes per kilogram the formula is

$$\text{cooking time (minutes)} = 16 \times \text{weight in kg}$$

The corresponding straight line graph, which starts from the point $(0, 0)$, is also drawn in Figure 52. In the microwave, a 3 kg chicken will take $16 \times 3 = 48$ minutes, so a second point on this new graph is $(3, 48)$.

The cooking times for a 1.6 kg chicken are 84 minutes (or 1 hour 24 minutes) in a conventional oven, and 26 minutes in the microwave.

## Activity 9

Here are some ideas. Electricity, gas and telephone bills in the UK normally include a charge per unit plus standing charges which you must pay even if you have used nothing else. These give straight-line relationships (assuming there are no special deals or charge bands) between use and charge which lead to graphs with intercepts and constant gradients.

In contrast, there is a directly proportional relationship between costs and quantity for any item bought on a cost-per-unit basis.

The relationship between the angle of the hour hand (measured from the top of a clock) and the time is a directly proportional relationship, over a 12-hour period. Similarly for the minute hand over 1 hour.

The quantity of wallpaper or paint needed to decorate a room is directly proportional to the area of the walls.

The distance travelled is directly proportional to travel time if you travel at a steady speed. It is also roughly proportional to the amount of petrol used. Conversely, travel time is directly proportional to distance if you travel at a steady speed.

## Activity 10

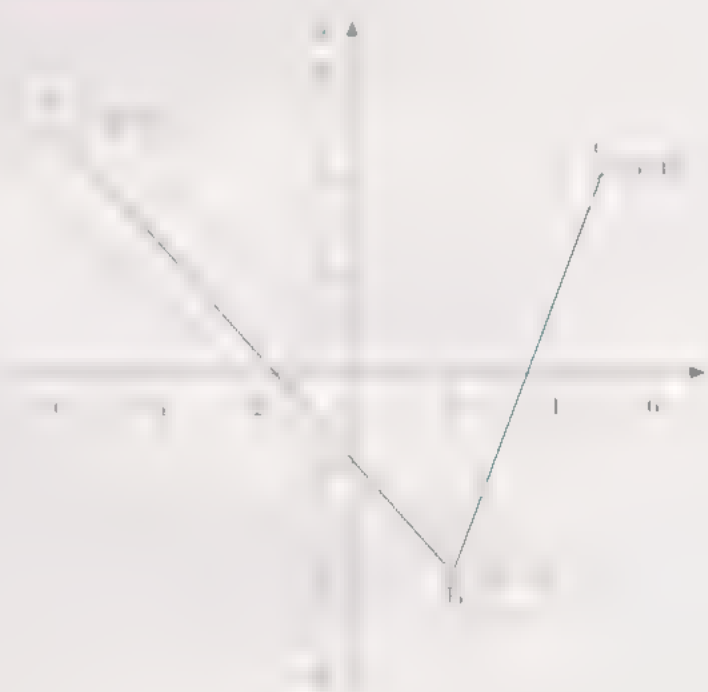

*Figure 53* Plot for Activity 10

For the line joining $A$ and $B$, the $x$-coordinate changes from $-6$ to $+2$, an increase of 8. The $y$-coordinate changes from 5 to $-4$, an 'increase' of $-9$. So the gradient is negative and equal to $(-9)/8 = -1.125$, and the line slopes *down* from left to right.

For the line from $B$ to $C$, the $x$-coordinate increases by 3, from 2 to 5. The $y$-coordinate increases by 8, from $-4$ to 4. So the gradient is positive and equal to $8/3 (\simeq 2.667)$, and the line slopes *up* from the left to right.

## Activity 11

Table 6 shows the completed data for the cubic relationship.

*Table 6* Data for cubic relationship

| [illegible] | [illegible] | [illegible] |
|---|---|---|
| $-3$ | $-27$ | $(-3, -27)$ |
| $-2$ | $-8$ | $(-2, -8)$ |
| $-1$ | $-1$ | $(-1, -1)$ |
| 0 | 0 | (0, 0) |
| 1 | 1 | (1, 1) |
| 2 | 8 | (2, 8) |
| 3 | 27 | (3, 27) |

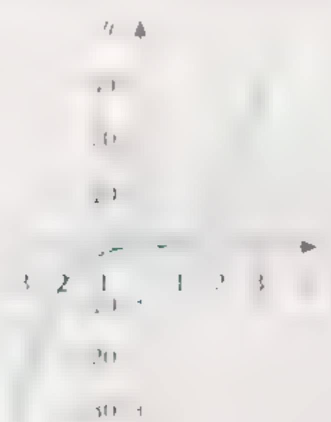

*Figure 54* Sketch of the cubic relationship

Figure 54 shows a sketch of the cubic relationship. Starting in the third quadrant you can see that the gradient of the curve is positive but the slope decreases as you move along the curve, closer to the point (0, 0), the origin of the graph. At the origin itself, the slope of the graph is zero. As the value of the $x$-coordinate becomes positive, the slope begins to increase again. The curve gets steeper as you move to the right in the first quadrant. It turns out that the slope of this cubic curve is always positive, except at the origin where the slope is zero.

## Activity 12

If you are to be influenced by the shape of the graphs, then clearly the numbers of doctors, dentists, nurses, and midwives rocketed over the period. Not only that, but both graphs are a lot steeper in 1987 than they were in 1978. In other words, the rate at which the numbers were increasing was itself going up. There will not just be a steady growth of doctors and nurses, it seems to suggest, there will be increasingly more and more as time goes on. A better health service indeed—the graphs seem to speak for themselves. And this phenomenal growth, people are clearly encouraged to think, has been achieved as a result of the support of the political party who produced the publicity postcards.

If you look carefully, however, you will see that a number of graphical devices have been used to create this overall impression. The first has been to use a relatively short horizontal axis, and to have a tall, narrow shape to emphasize the growth. Notice also, that unequal periods of time have been given equal space. Further, the graphs have been drawn using axes where the vertical scales do not start from zero. And, although it seems you are encouraged to compare the numbers of doctors and nurses because both graphs have been drawn in the same space—and perhaps even to see the numbers converging as the two graphs come closer and closer—in fact, the vertical scales of the two graphs are quite different.

Figure 55 shows the data redrawn. You can see that the effect is rather less dramatic.

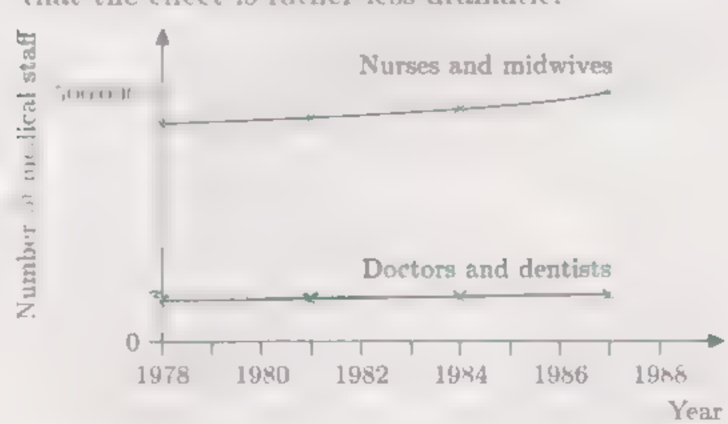


*Figure 55* Redrawing the health service data

## Activity 13

Napoleon's army reached Smolensk on the 14th of November 1812, with about 24 000 men, less than 6 per cent of the 422 000 who started the campaign.

The lowest temperature indicated by Minard is $-30\,°$R, on 6 December. The Celsius scale is directly proportional to the Réaumur temperature scale. The freezing point of water is the same on both scales, but the boiling point is different. On a conversion graph with degrees Réaumur plotted horizontally, and degrees Celsius plotted vertically, the two fixed points are $(0, 0)$ and $(80, 100)$. If you sketch the graph and work out the formula you should be able to convert between the two scales. It is

$$\text{degrees Celsius} = \tfrac{100}{80} \times \text{degrees Réaumur}$$

So a temperature of $-30\,°$R is equivalent to $-30 \times (100/80) = -37.5\,°$C.

The numbers on Minard's graphic indicate that 100 000 men left Moscow in October 1812 for the long march home. Only 20 000 were left by the time they were joined at Bobr (*Bobruysh*) by a 30 000 strong column from the north, boosting the army to 50 000 men. Of this number, two out of every five men had marched from Moscow. 22 000 were lost crossing the Berezina River in the sub-zero temperatures, and of the survivors only 4000 reached the Niemen River, their starting point six months earlier. Assuming that this number contained the same proportion of men, two out of five, who had left Moscow, then of that 100 000 strong force only 1600 were still with the army. Death and desertion accounted for the rest. Remember also that women and children would have accompanied the army, as well as others not directly part of the fighting force. Their numbers are not recorded, but we must assume that they shared the soldiers' fate.

Minard's graphic speaks eloquently and poignantly of a human tragedy of the highest proportions.

## Activity 14

There are no comments on this activity.

## Activity 15

(a) The formula for speed is $s = d/t$. 45 minutes is equal to 0.75 hours, so the average speed is $25/0.75 = 33.3$ km per hour.

(b) The formula for distance is $d = s \times t$. In this case, $s$ is 75 km per hour and $t$ is $30/60 = 0.5$ hours. The distance is $75 \times 0.5 = 37.5$ km.

(c) The formula for travel time is $t = d/s$. The time to cover 500 metres at 10 metres per second is $500/10 = 50$ seconds.

## Activity 16

To tackle this activity, you need to be able to estimate Rana's walking speed as she completes her round. Her speed at any point is indicated by the slope of the distance–time graph. A steep slope means she was walking quickly and covered the distance in a relatively short time. A gentle slope means she was walking relatively slowly and took longer to cover the distance. A slope of zero—where the graph is horizontal—represents zero speed, indicating that she had stopped.

Rana was walking fastest on section $BC$. Here, the slope of the graph is steeper than in any other section, indicating the highest speed.

Section $CD$ is horizontal indicating Rana's speed was zero for a short time. She takes the break at the furthest distance from the start.

Assuming Rana walks slowest when she is going up the hill, this section is represented by $AB$.

Section $DE$ represents Rana's return. The gradient is negative because the distance from the start is decreasing with time, as Rana completes the round and returns back along the same route to her starting point.

## Activity 17

The distance travelled after lunch is 2 km at 10 km per hour, which takes 0.2 hour, or 12 minutes. The total distance from the start is 7 km.

The playground stop is 3 km from the cycle hire shop, and hence 4 km back from the furthest point on the ride. At an average speed of 10 km per hour, it will take 0.4 hours, or 24 minutes to reach the playground.

Table 7 shows all the distances and times for the ride.

*Table 7* Completed distance and time data

| Time from start/minutes | Distance from start/km | Comments |
|---|---|---|
| 0 | 0 | leave cycle-hire shop |
| 25 | 5 | reach pub |
| 70 | 5 | leave pub |
| 82 | 7 | short break |
| 97 | 7 | start back |
| 121 | 3 | playground stop |
| 151 | 3 | leave playground |
| 166 | 0 | arrive at cycle-hire shop |

## Activity 18


*Figure 56* Complete distance–time graph for the cycle ride

The distance–time graph for the complete cycle ride is shown in Figure 56. The 7 km journey back takes $166 - 97 = 69$ minutes (including the stop at the playground). So the average speed is $(7/69) \times 60 = 6.1$ km per hour. This speed is represented by the slope of the dashed line on the graph. The line has a negative gradient, indicating that the direction of the journey is back towards the starting point.

## Activity 19

Figure 39(a) suggests a period of no movement, followed by an instantaneous change in distance, followed again by no movement. This distance–time profile (the general shape is sometimes called a 'step change') is impossible. All movements take some time to complete, you cannot travel a distance in no time at all.

In Figure 39(b), the journey starts at a steady speed. This is followed by a brief period during which the speed is zero, and hence the slope of the line is zero. The next section suggests that a further distance is travelled, but time appears to be reversed, so that the traveller arrives back at the same time they started, but at a different place. Fine for time travellers, but impossible in practice.

Figure 39(c) shows a perfectly reasonable distance–time graph. The journey starts at a constant speed and distance increases with time. The direction of travel then changes, and the traveller begins to return towards the starting point at a steady speed. Direction changes again and the graph indicates a steady speed away from the starting point during the final section.

## Activity 20

At 1.30 pm Alice and Bob will be about 80 km apart.

They will pass at about 265 km north of Milton Keynes, at about 2.10 pm.

## Activity 21

(a) To arrive at the same time as Bob, Alice must cover the 105 km in 90 minutes, or 1.5 hours. So her average speed must be $105/1.5 = 70$ km per hour.

(b) Alice covers her journey of 340 km in 6 hours, so her average speed (overall) is $340/6 = 56.7$ km per hour. Bob completes his journey of 420 km in 7 hours 50 minutes, or 7.83 hours. So his average speed (overall) is $420/7.83 = 53.6$ km per hour.

## Activity 22

Diagrams are often helpful in making ideas easier to understand. In planning your series of diagrams, you needed to show the particular properties and characteristics of distance–time graphs and how you use them to take readings. As your audience may not have met such graphs before, your diagrams should be clear and labelled appropriately. You may need to limit your use of technical language or explain it. Each diagram could, for instance, demonstrate a particular feature, until the final diagram is a completed graph perhaps with an example showing how to use it.

Your diagrams should include the following elements developed in a logical order.

- ◇ A vertical axis and a horizontal axis that are both labelled, including units. The vertical axis should give the distance from a specified place (in a specified direction) at a time shown on the horizontal axis (after a specified time).
- ◇ The journey points are correctly plotted on the graph, are labelled and joined with straight lines.
- ◇ A title for the graph.
- ◇ The example needs to demonstrate that the slope of the graph gives the average speed between two places: the steeper the slope, the faster the average speed. Parts of the graph sloping upwards/downwards mean travel is away from/towards the starting point.

## Activity 23


*Figure 57* Distance–time graph of the train journey

The highest speed is reached on the section between Hampton Loade and Highley. This is where the slope of the distance–time graph is steepest. The 3.3 km section was covered in 7 minutes. So the average speed is $3.3/7 = 0.47$ km per minute, or about 28 km per hour.

The graph indicates that the lowest speed occurs on the section between Bewdley and Kidderminster. Here, the slope of the graph is least. The train covers the 4.5 km in 20 minutes, an average speed of $4.5/20 = 0.225$ km per minute, or 13.5 km per hour.

The complete journey of 24.5 km is covered in 70 minutes. So the average speed is $24.5/70 = 0.35$ km per minute, or 21 km per hour.

## Activity 24

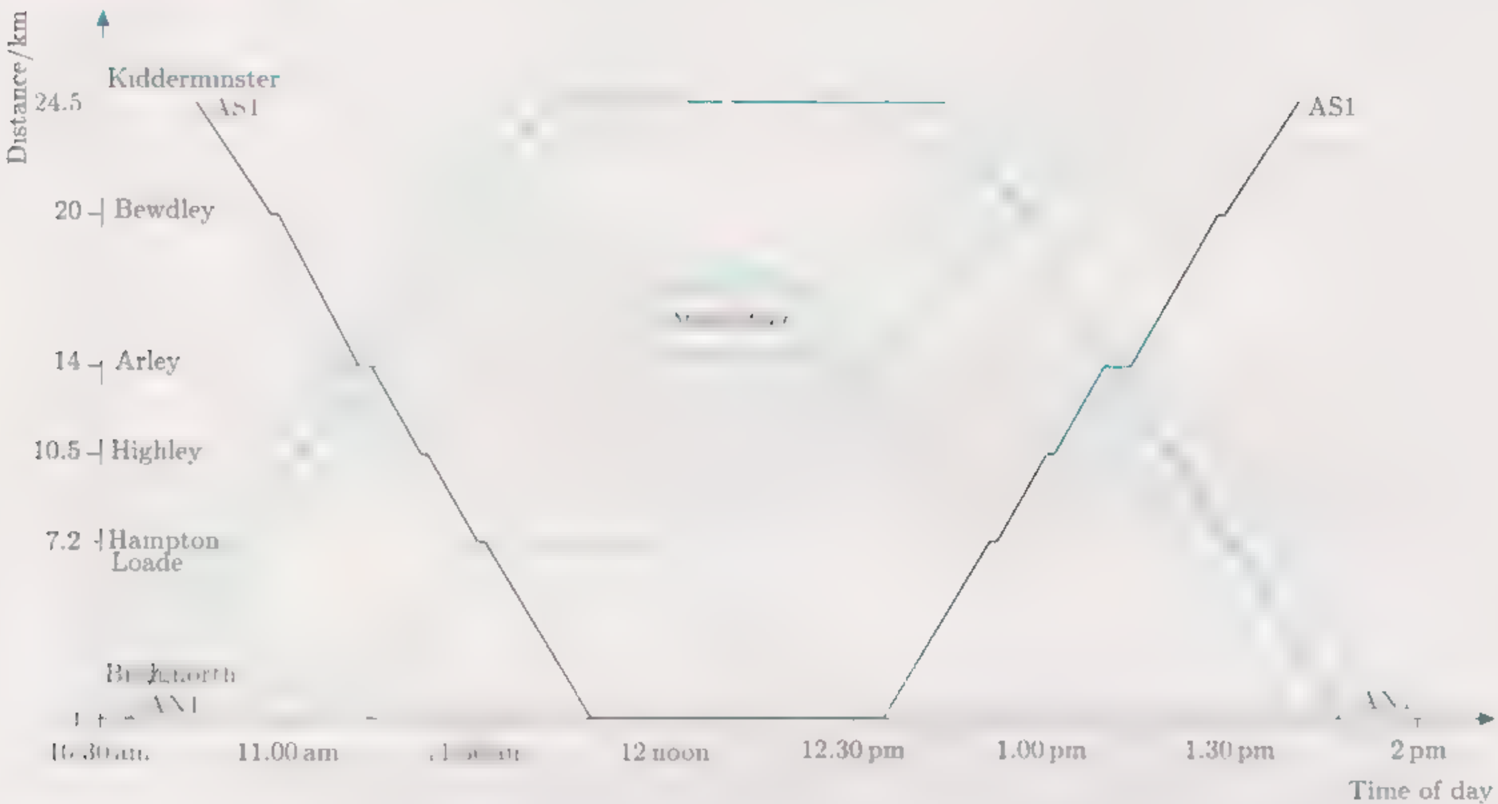


*Figure 58* Distance–time graph of trains AN1 and AS1

If AS1 is on time, train AN1 waits at Arley for 2 minutes. If it travelled at 35 km per hour, it would cover the 6 km from Bewdley in $6/35 = 0.17$ hours, or about 10 minutes, arriving at Arley at 1.08 pm. It would then have to wait 5 minutes until it could depart on schedule at 1.13 pm.

## Activity 25

(a) To cross the service train without stopping, it should pass through Bewdley just after AN1 arrives at 12.56 am. Scheduling the cross at 12.57 would leave a little time for AN1 to arrive.

(b) If the train leaves Hampton Loade at 12.00 and passes through Bewdley at 12.57, it will have covered 12.8 km in 57 minutes. Its average speed will be $(12.8/57) \times 60 = 13.5$ km per hour.

(c) From the graph or by calculation, the train should reach Kidderminster just after 1.16 pm.

## Activity 26

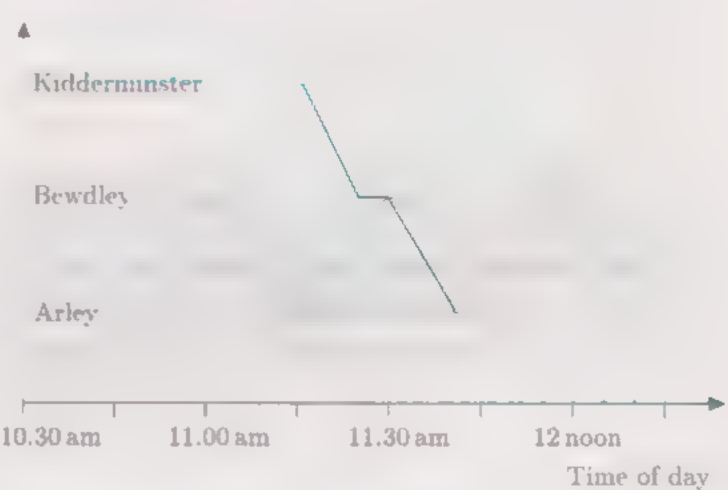


*Figure 59* Graphs of the Santa Specials

Figure 59 shows the graphs for all the Santa Specials between the given times. Notice that the graph for the 11.15 from Kidderminster is the same as the graph for the 10.45, only shifted along by thirty minutes. Similarly, the 11.46 arrival at Kidderminster has the same graph as the 11.16 arrival, but shifted forward by thirty minutes.

(a) The 11.15 from Kidderminster arrives at Bewdley at 11.25 am, and departs at 11.30 am.

(b) It will arrive at Arley at 11.42 am.

(c) The 11.46 arrival at Kidderminster arrives at Bewdley at 11.30 am and leaves Bewdley at 11.35 am.

(d) It leaves Arley at 11.16 am.

(e) To run a fifteen-minute service, extra trains would have to leave Kidderminster at 11.00 am (and 11.30 am). But these trains would have to cross the 11.05 (and 11.35) trains from Bewdley. Since there is no crossing place between Bewdley and Kidderminster, the extra trains cannot run.

## Activity 27

Passengers are likely to find timetables more useful. Signalling and track staff will probably find both helpful. The graph shows trains in both directions at once at intermediate spots, but the times at stations are easier to read from the table. Planning staff need graphs to be able to fit in a new service.

## Activity 28

There are no comments for this activity.

## Acknowledgements

Grateful acknowledgement is made to the following sources for permission to reproduce material in this unit:

### Illustrations

p. 7, Figure 1: *Guardian*, 12.2.1994; p. 8, Figure 2: *Guardian*, 6.5.1995, Figure 3: *Guardian*, 11.5.95; p. 30, Figure 21 and p. 31, Figure 22: William Playfair, *The Commercial and Political Atlas* (1786), photograph by permission of the British Library; p. 33, Figure 24: from 'Fixed-rate loans shoot up' by Diana Wright from *The Sunday Times*, 15.5.1994 © Times Newspapers Limited 1994; p. 34, Figure 26: *Guardian*, 10.1.1994; p. 55, Figure 43 and p. 56, Figure 44: E. J. Marey, *La Méthode Graphique* (1885), photograph by permission of the British Library; p. 59, Figure 42: by courtesy of the Severn Valley Railway Company.

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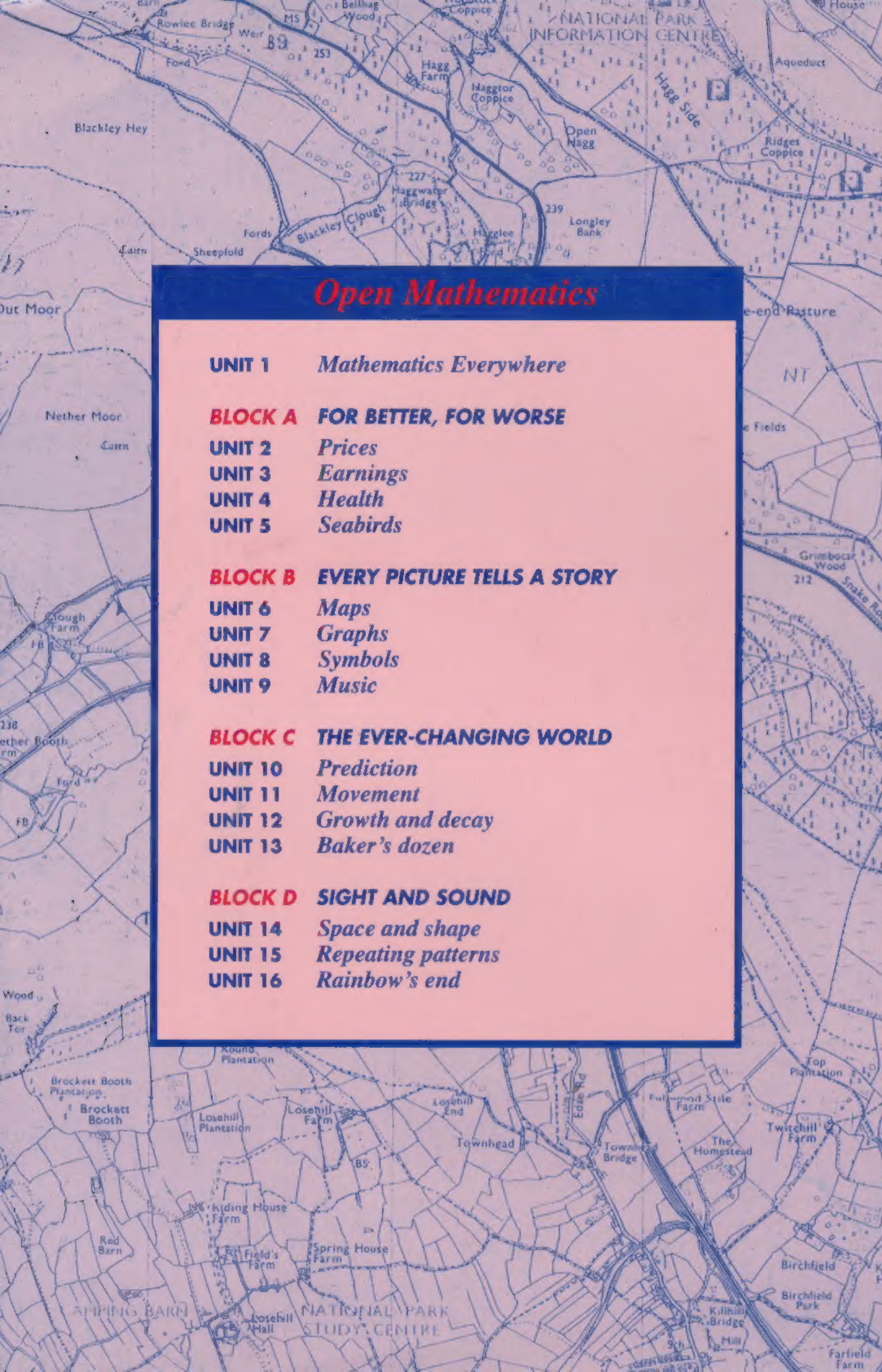


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